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MAXIMUM ANGLE CONDITION IN THE CASE OF SOME NONLINEAR ELLIPTIC PROBLEMS

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Abstract

In this paper the finite element method is analyzed for nonlinear elliptic variational problem which is formally equivalent to a two-dimensional nonlinear elliptic boundary problem with mixed nonhomogeneous boundary conditions. The given problem is analyzed under the maximum angle condition and is solved in the case of a bounded domain Ω whose boundary $\partial\Omega$ consists of two circles Γ_1, Γ_2 of the same centre S_0. These circles have the radii R_1, $R_2 = R_1 + \varrho$, where $\varrho \ll R_1$. The finite element analysis is restricted to the case of semiregular finite elements with polynomials of the first degree. At the end some numerical results are introduced.

Introduction

The theory presented generalizes the results obtained in [2] and [12]. In [2] the problem is formulated on an arbitrary domain with a Lipschitz-continuous boundary and the finite element method is analyzed under the minimum angle condition. In [12] the finite element method is analyzed for a linear strongly elliptic mixed boundary value problem under the maximum angle condition. In this paper we consider the same domain as in [12] but the problem is nonlinear. Our assumptions concern the boundary, the data and the form $a(u, v)$, which is nonlinear in u and linear in v. We prove the convergence of approximate solutions to the exact solution u under the condition $u \in H^1(\Omega)$. The theory is briefly made in [14] and precisely in [3]. Some numerical results are discussed at the end.

In [11] the finite element method for a special monotone problem, which has applications in magnetostatics, was analyzed under the maximum angle condition. The results can be considered to be a special case of the present text.

The notation of Sobolev spaces, their norms and seminorms is the same as in [5].

1. Formulation of the problem

1.1. **Boundary value problem.** We will consider the boundary value problem

$$-\sum_{i=1}^{2} \frac{\partial b_i}{\partial x_i}(\cdot, u, \nabla u) + b_0(\cdot, u, \nabla u) = f(x)\,, \quad x \in \Omega\,, \tag{1}$$

$$u = 0 \quad \text{on } \Gamma_1\,, \tag{2}$$

$$\sum_{i=1}^{2} b_i(\cdot, u, \nabla u) n_i(\Omega) = q \quad \text{on } \Gamma_2 \tag{3}$$

where Ω (see Fig. 1) is a two-dimensional bounded domain with a boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$, Γ_1 and Γ_2 being circles with radii R_1 and $R_2 = R_1 + \varrho$, respectively. We assume that the

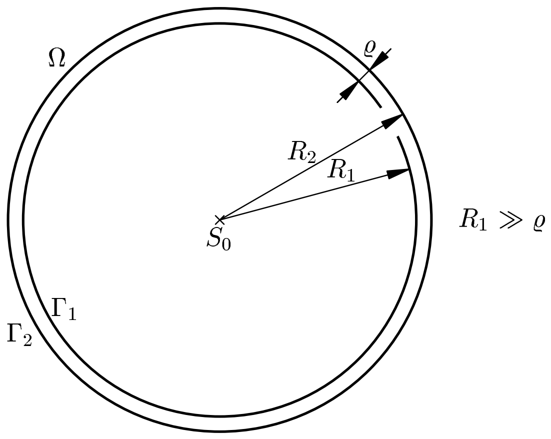

Рис. 1: The domain Ω

circles Γ_1, Γ_2 have the same center S_0 and that $R_1 \gg \varrho$. Obviously, $\partial\Omega$ is Lipschitz continuous. The symbols $n_i(\Omega)$ $(i = 1, 2)$ denote the components of the unit outward normal to $\partial\Omega$. Further, $f : \Omega \to \mathbb{R}^1$, $b_i : \Omega \times \mathbb{R}^3 \to \mathbb{R}^1$ (i.e., $b_i = b_i(x, \xi) = b_i(\cdot, u, \nabla u)$, where $x = (x_1, x_2) \in \Omega$, $\xi = (\xi_0, \xi_1, \xi_2) = (u(x), \nabla u(x)) \in \mathbb{R}^3$, i = 0, 1, 2) are given functions and $\nabla u = (\partial u/\partial x_1, \partial u/\partial x_2)$.

We solve this type of elliptic boundary problem by an almost standard finite element method.

1.2. **Weak formulation.** We will use the Lebesgue spaces $L_2(\Omega)$, $L_2(\partial\Omega)$, $L_\infty(\Omega)$ and the Sobolev spaces $H^1(\Omega)$, $H^2(\Omega)$, $W^{1,\infty}(\Omega)$ equipped with their usual norms $\|\cdot\|_{0,\Omega}$, $\|\cdot\|_{0,\partial\Omega}$, $\|\cdot\|_{0,\infty,\Omega}$ and $\|\cdot\|_{1,\Omega}$, $\|\cdot\|_{2,\Omega}$, $\|\cdot\|_{1,\infty,\Omega}$, respectively (see [1, 5, 7]). The seminorms in the spaces $H^1(\Omega)$ and $H^2(\Omega)$ will be denoted by $|\cdot|_{1,\Omega}$ and $|\cdot|_{2,\Omega}$, respectively.

1. Assumptions. Let $\{\Omega_h\}$ $(h \in (0, h_0))$ be a set of polygonal approximations of Ω. Let $\widetilde{\Omega} \subset \mathbb{R}^2$ be a bounded domain such that

$$\widetilde{\Omega} \supset \overline{\Omega} \cup \overline{\Omega}_h \quad \forall h \in (0, h_0)\,. \tag{4}$$

Let the functions $f : \widetilde{\Omega} \to \mathbb{R}^1$, $q : \Gamma_2 \to \mathbb{R}^1$ and $b_i : \widetilde{\Omega} \times \mathbb{R}^3 \to \mathbb{R}^1$ $(i = 0, 1, 2)$ have the following properties:

(A) (Growth condition) The functions $b_i(x, \xi)$ $(x \in \widetilde{\Omega}, \xi = (\xi_0, \xi_1, \xi_2) \in \mathbb{R}^3)$ are continuous in $\widetilde{\Omega} \times \mathbb{R}^3$. There exists a constant $C > 0$ such that

$$|b_i(x, \xi)| \le C\Big(1 + \sum_{j=0}^{2} |\xi_j|\Big) \quad \forall x \in \widetilde{\Omega}\,, \quad \forall \xi \in \mathbb{R}^3 \quad (i = 0, 1, 2)\,.$$

(B) The derivatives $(\partial b_i/\partial \xi_j)(x,\xi)$ $(i,j=0,1,2)$ are continuous and bounded in $\widetilde{\Omega}\times\mathbb{R}^3$:

$$\left|\frac{\partial b_i}{\partial \xi_j}(x,\xi)\right| \le C \quad \forall x\in\widetilde{\Omega}\,, \quad \forall \xi\in\mathbb{R}^3\,.$$

(C) The functions b_i satisfy

$$\sum_{i,j=0}^{2}\frac{\partial b_i}{\partial \xi_j}(x,\xi)\eta_i\eta_j \ge \alpha\sum_{i=1}^{2}\eta_i^2 \quad \forall x\in\widetilde{\Omega}\,, \quad \forall \xi,\eta\in\mathbb{R}^3$$

where $\alpha>0$ is a constant independent of x,ξ and η.

(D) The functions $\partial b_i/\partial x_j$ $(i=0,1,2; j=1,2)$ are continuous in $\widetilde{\Omega}\times\mathbb{R}^3$. There exists a constant $C>0$ such that

$$\left|\frac{\partial b_i}{\partial x_j}(x,\xi)\right| \le C\Big(1+\sum_{j=0}^{2}|\xi_j|\Big) \quad \forall x\in\widetilde{\Omega}\,, \quad \forall \xi\in\mathbb{R}^3 \quad (i\ =0,1,2; j=1,2)\,.$$

(E) a) $f\in W^{1,\infty}(\widetilde{\Omega})$;
b) $\overline{u}=z$ on Γ_1, where $z\in W_p^1(\Omega)$ $(p>2)$;
c) the function q defined on $\Gamma_2=\partial\Omega-\overline{\Gamma}_1$ is piecewise of class C^2,

$$q\in PC^1(\Gamma_2)\,.$$

(F) The quadrature formulas which will be used in the disertation have the forms

$$\int_T F(x_1,x_2)\mathrm{d}x \approx \mathrm{meas}_2 T\sum_{i=1}^{I}\omega_{T,i}F(x_{T,i})\,, \tag{5}$$

$$\int_{\Gamma_{2h}} F(x)\mathrm{d}s = \sum_{S_h}\int_{S_h} F(x)\mathrm{d}s \approx \sum_{S_h} s_h\sum_{i=1}^{I}\beta_i F(x_{T,i}) \tag{6}$$

with degrees of precision $d=1$ and we use only such formulas which satisfy

$$\omega_{T,i}>0 \quad (i=1,\dots,I), \qquad \sum_{i=1}^{I}\omega_{T,i}=1.$$

($S_h\in\partial\Omega_h$ is the corresponding side of the triangle $T\in\mathscr{T}_h$ which approximates T^{id} (so called ideal triangle) and s_h is the length of the side S_h.)

A weak solution of problem (1)–(3) is a solution of the following variational problem (which can be obtained from (1)–(3) by means of Green's theorem in a standard way).

2. Problem (Continuous). Let us set

$$V\ = \{v\in H^1(\Omega): v\ =0 \text{ on } \Gamma_1\}\,, \tag{7}$$

$$a(w,v) = \int_\Omega \left(\sum_{i=1}^{2} b_i(\cdot,w,\nabla w)\frac{\partial v}{\partial x_i} + b_0(\cdot,w,\nabla w)v\right)\mathrm{d}x \quad \forall w,v\ \in H^1(\Omega)\,, \tag{8}$$

$$L(v) = L^{\Omega}(v) + L^{\Gamma}(v) = \int\limits_{\Omega} vf \, \mathrm{d}x + \int\limits_{\Gamma_2} vq \, \mathrm{d}s \,. \tag{9}$$

Find $u \in V$ such that

$$a(u,v) = L(v) \quad \forall v \ \in V \ . \tag{10}$$

If $b_i(x,\xi) = k_i(x)\xi_i, (i = 0,1,2)$, then Continuous Problem 2 is reduced to a linear problem. In this case assumptions (A)–(E) can be easily satisfied. In Example 3 we give example of functions $b_i(x,\xi)$ which satisfy (A)–(E) and which do not reduce Continuous Problem 2 to a linear one.

3. Example. Let $\nu = \nu(x,s)$ $(x \in \widetilde{\Omega}, s \in \langle 0,\infty))$ be the function with the following properties:

a) $\nu(x,s)$ and the derivatives $\partial\nu/\partial s$ are continuous in $\widetilde{\Omega} \times \langle 0,\infty)$.

b) The derivatives $\partial\nu/\partial x_j$ $(j = 1,2)$ are continuous in $\widetilde{\Omega} \times \langle 0,\infty)$ and bounded:

$$\left|\frac{\partial\nu}{\partial x_j}(x,s)\right| \le C, \quad (x,s) \in \widetilde{\Omega} \times \langle 0,\infty) \quad (j = 1,2)\,. \tag{11}$$

c) There exists constants $0 < c_1 < c_2$ such that

$$c_1 \le \frac{\partial}{\partial s}(s\nu(x,s)) \le c_2, \quad \forall (x,s) \in \widetilde{\Omega} \times \langle 0,\infty)\,. \tag{12}$$

Assumption c) has an important consequence: if we integrate (12) in $\langle 0,t\rangle$ $(t > 0)$ then we obtain

$$c_1 \le \nu(x,t) \le c_2, \quad \forall (x,s) \in \widetilde{\Omega} \times (0,\infty)\,.$$

This result and assumption a) give

$$c_1 \le \nu(x,t) \le c_2, \quad \forall (x,s) \in \widetilde{\Omega} \times \langle 0,\infty)\,. \tag{13}$$

We define

$$\begin{aligned} b_i(x,\xi) &:= \nu\big(x,(\xi_1^2+\xi_2^2)^{1/2}\big)\xi_i \quad (i = 1,2)\,, \\ b_0(x,\xi) &\equiv 0\,. \end{aligned} \tag{14}$$

Using assumptions a)–c) and relation (13) we can prove after some computations that the functions (14) satisfy Assumptions (B)–(E).

Functions (14) are used in the theory of magnetic fields. See for example [6].

The formal equivalence of boundary value problem (1)–(3) and variational problem (7)–(10) follows from Lemma 4 which is proved in [3].

4. Lemma. *Let a solution $u \in V$ of Problem 2 satisfy $u \in H^2(\Omega)$. Then relation (1) holds almost everywhere in Ω and relation (3) holds almost everywhere on Γ_2.*

To be able to solve Continuous Problem 2 by the finite element in its simplest form let us approximate the domain Ω by a polygonal domain Ω_h with the polygonal boundary $\partial\Omega_h$. The vertices of the polygonal boundary $\partial\Omega_h$ lies on $\partial\Omega$. The symbols Γ_{1h} and Γ_{2h} denotes the parts of $\partial\Omega_h$ approximating Γ_1 and Γ_2, respectively (see Fig. 2).

We have committed by it the first variational crime.

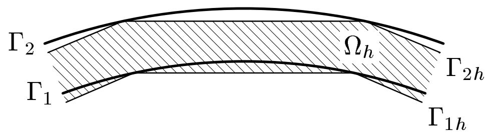

Рис. 2: A domain Ω_h

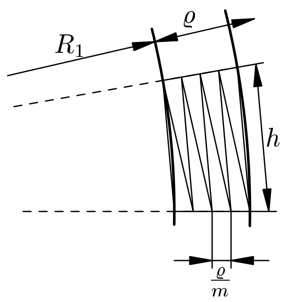

Рис. 3: The partition

We admit to use narrow triangles. This means that we will have

$$\frac{\varrho}{m} \ll h \tag{15}$$

in our considerations, where h is the length of the greatest segment and ϱ/m is the length of the shortest segment in the partition of Ω_h (see Fig. 3). The corresponding partition consisting of closed triangles $\overline{T}$ will be denoted by $\mathscr{T}_h$.

5. Definition.

a) We say that the set $\{\mathscr{T}_h\}$ of triangulations with $h \to 0$ satisfies the *maximum angle condition* if there exists a positive constant $\gamma_0 < \pi$ such that

$$\gamma_T \le \gamma_0 < \pi \quad \forall \overline{T} \in \mathscr{T}_h, \quad \forall \mathscr{T}_h \in \{\mathscr{T}_h\}\,, \tag{16}$$

where γ_T is the magnitude of the maximum angle of the triangle T.

b) We say that the set $\{\mathscr{T}_h\}$ of triangulations with $h \to 0$ satisfies the *minimum angle condition* if there exists a positive constant $\vartheta_0 > 0$ such that

$$\vartheta_T \ge \vartheta_0 > 0 \quad \forall \overline{T} \in \mathscr{T}_h, \quad \forall \mathscr{T}_h \in \{\mathscr{T}_h\}\,, \tag{17}$$

where ϑ_T is the magnitude of the minimum angle of the triangle T.

(In case a) we can have $\alpha \to 0$, which is impossible in case b).)

c) Irregular triangle can have two angles arbitrarily small.

We covered Ω_h by semiregular triangles, i.e., triangles satisfying the maximum angle condition (see Definition 5 and Fig. 3). If the triangulation satisfies the minimum angle condition the maximum angle condition is satisfied too. Now we will approximate the spaces $H^1(\Omega)$ and V.

1.3. **Discrete problem.** We define spaces

$$X_h = \{v \in C(\overline{\Omega}_h) : v\,|_{\overline{T}} \text{ is a linear polynomial } \forall \overline{T} \in \mathscr{T}_h\} \tag{18}$$

and

$$V_h = \{v \in X_h : v = 0 \text{ on } \Gamma_{1h}\} \equiv \{v \in X_h : v(P_i) = 0 \quad \forall P_i \in \Gamma_1\}\,, \tag{19}$$

P_i being the nodes of the given triangulation lying on Γ_1.

We set

$$\widetilde{a}_h(w,v) = \int\limits_{\Omega_h} \left(\sum_{i=1}^{2} b_i(\cdot, w, \nabla w) \frac{\partial v}{\partial x_i} + b_0(\cdot, w, \nabla w) v \right) \mathrm{d}x \quad \forall w, v \in H^1(\Omega_h) \tag{20}$$

and

$$\widetilde{L}_h^{\Omega}(v) = \int\limits_{\Omega_h} v f \,\mathrm{d}x \quad \forall v \in X_h\,. \tag{21}$$

To define $\widetilde{L}_h^{\Gamma}(v)$ is more complicated and we refer to [12] or [3].

The symbols $a_h(w,v), L_h^{\Omega}(v)$ and $L_h^{\Gamma}(v)$, where $w, v \in X_h$, will denote the approximations of $\widetilde{a}_h(w,v), \widetilde{L}_h^{\Omega}(v)$ and $\widetilde{L}_h^{\Gamma}(v)$, respectively, by means of numerical integration. We can write the results by the forms

$$\begin{aligned} a_h(v,w) = \sum_{T \in \mathscr{T}_h} \mathrm{meas}_2 T \Bigg(\sum_{i=1}^{2} \frac{\partial w}{\partial x_i}\bigg|_T \sum_{j=1}^{I} 2\omega_{T_0,j} b_i(x_{T,j}, v(x_{T,j}), \nabla v|_T) + \\ + w(x_{T,j}) \sum_{j=1}^{I} 2\omega_{T_0,j} b_0(x_{T,j}, v(x_{T,j}), \nabla v|_T) \Bigg)\,, \end{aligned} \tag{22}$$

$$L_h(v) = L_h^{\Omega}(v) + L_h^{\Gamma}(v)\,, \tag{23}$$

where

$$L_h^{\Omega}(v) = \sum_{T \in \mathscr{T}_h} \mathrm{meas}_2 T \sum_{j=1}^{I} 2\omega_{T_0,j} f(x_{T,j}) v(x_{T,j})\,, \tag{24}$$

$$L_h^{\Gamma}(v) = \sum_{S_h \subset \Gamma_{2h}} s_h \sum_{j=1}^{I} \beta_i q_h(x_{T,j}) v(x_{T,j})\,.$$

Numerical integration is not exact. Thus we have committed by it the second variational crime.

Now we define a finite element discrete problem for the solution of Problem 2 with the use of numerical integration.

6. Problem (Discrete). Find $u_h \in V_h$ such that

$$a_h(u_h, v) = L_h(v) \quad \forall v \ \in V_h\,. \tag{25}$$

Let us note that X_h and V_h are finite dimensional approximations of $H^1(\Omega)$ and V, respectively. Using them we have committed the third and last variational crime.

In order to prove Theorem 10 (our Abstract Error Estimate) we must use discrete Friedrichs' inequality in the case of narrow triangles satisfying the maximum angle condition in the following form (this Lemma is proved in [14]).

7. Lemma (Discrete Friedrichs' inequality). *We have*

$$\|v\|_{1,\Omega_h} \le C|v|_{1,\Omega_h}, \quad \forall v \in V_h,\, \forall h < h_0\,, \tag{26}$$

where the constant C does not depend on h and v.

Since we use the partition of the domain Ω_h which is overlapping the origin domain Ω we need to use an extension of the function u. It is described in the following Lemma 8 which is proved in [3].

8. Lemma. *Let Ω be a two-dimensional bounded domain with the boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$ where Γ_1 and Γ_2 are circles with radii R_1 and $R_2 = R_1 + \varrho_0$. We assume that the circles Γ_1 and Γ_2 have the same center S_0 and that*

$$R_1 \gg \varrho_0\,. \tag{27}$$

Let Γ_0 be the circle with a center S_0 and radius $R_0 = R_1 - \varrho$ and let $\widetilde{\Omega}$ be a bounded domain such that $\partial\widetilde{\Omega} = \Gamma_0 \cup \Gamma_2$. Then there exists a linear and bounded extension operator (of Nikolskij-Hestenes type) $\varepsilon_2 : H^2(\Omega) \to H^2(\widetilde{\Omega})$ with property $\varepsilon_2 : H^1(\Omega) \to H^1(\widetilde{\Omega})$ and such that the constant C appearing in the inequality

$$\|\varepsilon_2(v)\|_{2,\widetilde{\Omega}} \le C\|v\|_{2,\Omega}\,, \quad \forall v \in H^2(\Omega) \tag{28}$$

does not depend on R_1/ϱ.

9. Remark. By the standard way can be proved that Continuous Problem 2 has a unique solution and that the Discrete Problem 6 has a unique solution. The prove is made in [3].

2. An abstract error estimate

10. Theorem (Abstract Error Estimate). *Let Assumptions 1 be satisfied. For all $h \in (0, h_0)$ we have*

$$\begin{aligned} C^{-1}\|\widetilde{u} - u_h\|_{1,\Omega_h} \le \inf_{v\in V_h} \|v - \widetilde{u}\|_{1,\Omega_h} + \inf_{v\in V_h} \sup_{\substack{w\in V_h\\ w\neq 0}} \frac{|a_h(v,w) - \widetilde{a}_h(v,w)|}{\|w\|_{1,\Omega_h}} \\ + \sup_{\substack{w\in V_h\\ w\neq 0}} \frac{|\widetilde{L}_h^{\Omega}(w) - L_h^{\Omega}(w)|}{\|w\|_{1,\Omega_h}} + \sup_{\substack{w\in V_h\\ w\neq 0}} \frac{|\widetilde{L}_h^{\Gamma}(w) - L_h^{\Gamma}(w)|}{\|w\|_{1,\Omega_h}} + \sup_{\substack{w\in V_h\\ w\neq 0}} \frac{|\widetilde{a}_h(\widetilde{u},w) - \widetilde{L}_h(w)|}{\|w\|_{1,\Omega_h}}\,, \end{aligned} \tag{29}$$

where C is a positive constant not depending on the solution $u \in H^1(\Omega)$ of Continuous Problem 2, $u_h \in V_h$ is the solution of Discrete Problem 6 and $\widetilde{u} = E(u)$ with $E : H^1(\Omega) \to H^1(\widetilde{\Omega})$.

The proof of the Theorem 10 we can find in [3].

The estimating of the terms on the right side of the inequality (29) consists of the three separated parts. The first part is focused to the estimating of the first term on the right side of inequality (29) which expresses the error of the interpolation (see Section 5, [3]). The second part covers the estimations related with the numerical integration. Errors of the numerical integration are included in the second, the third and the fourth term of the inequality (29). For details se Section 6, [3]. The last term on the right side of the inequality (29) represents the error of the approximation of the boundary Ω by Ω_h and estimations are precisely made in Section 7, [3].

We introduce only the results now:

11. Theorem. *We have*

$$\inf_{v \in V_h} \|v - \widetilde{u}\|_{1,\Omega_h} \leq Ch\|u\|_{2,\Omega}, \tag{30}$$

where the constant C is independent of h, u the triangulation $\mathscr{T}_h$, and $\widetilde{u} \in H^2(\widetilde{\Omega})$.

12. Theorem.

$$IS := \inf_{v \in V_h} \sup_{w \in V_h, w \neq 0} \frac{|a_h(v,w) - \widetilde{a}_h(v,w)|}{\|w\|_{1,\Omega}} \leq Ch(1 + \|u\|_{1,\Omega}), \tag{31}$$

where $u \in H^1(\Omega)$ is the solution of the Continuous variational Problem 2 and the constant C does not depend on h and u.

13. Theorem. *Let the degree of precision of a given quadrature formula be $d = 0$. Then we have*

$$\sup_{\substack{w \in V_h \\ w \neq 0}} \frac{|\widetilde{L}_h^{\Omega}(w) - L_h^{\Omega}(w)|}{\|w\|_{1,\Omega_h}} \leq Ch\sqrt{\mathrm{meas}_2 \Omega}\|f\|_{1,\infty,\widetilde{\Omega}} \,. \tag{32}$$

14. Theorem. *Let the degree of precision of quadrature formulas be $d \geq 1$. Then we have*

$$\sup_{\substack{w \in V_h \\ w \neq 0}} \frac{|\widetilde{L}_h^{\Gamma}(w) - L_h^{\Gamma}(\overline{w})|}{\|w\|_{1,\Omega_h}} \leq \frac{C}{\sqrt{\varrho}} h^d M_d(q)\sqrt{\mathrm{meas}_1 \Gamma_2} \tag{33}$$

where the constant C does not depend on q, ϱ and h and where in the most used case $d = 2$ we have

$$M_2(q) = 8 \max\left(\frac{3}{2}, \frac{5}{4}\frac{1}{R_2}\right) \max_{(x,y) \in \Gamma_2} \left(\left|\frac{\partial^2 q}{\partial x^2}\right|, \left|\frac{\partial^2 q}{\partial y^2}\right|, \left|\frac{\partial^2 q}{\partial x \partial y}\right|, \left|\frac{\partial q}{\partial x}\right|, \left|\frac{\partial q}{\partial y}\right|\right).$$

Thus, if we want to obtain the rate of convergence $O(h)$ we must assume that

$$C_1 h^2 \leq \frac{\varrho}{m} \quad (C_1 > 0)\,. \tag{34}$$

Assumption (34) is a restriction for semiregular triangles because $\frac{\varrho}{m}$ is the length of the smallest side of triangles in our triangulation $\mathcal{T}_h$.

15. Theorem. *Let $u \in H^2(\Omega)$ and let condition (34) be satisfied. Then*

$$\sup_{w\in V_h, w\neq 0} \frac{\left|\widetilde{a}_h(\widetilde{u}, w) - \widetilde{L}_h(w)\right|}{\|w\|_{1,\Omega_h}} \leq \frac{C}{\sqrt{\varrho}}h \tag{35}$$

where the constant C does not depend on ϱ, u, m, h and the triangulation $\mathcal{T}_h$.

If in addition condition

$$u \in H^2(\Omega) \cap W^{1,\infty}(\Omega) \tag{36}$$

is satisfied, i.e., $u \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$, then

$$\sup_{w\in V_h, w\neq 0} \frac{\left|\widetilde{a}_h(\widetilde{u}, w) - \widetilde{L}_h(w)\right|}{\|w\|_{1,\Omega_h}} \leq Ch \tag{37}$$

where again the constant C does not depend on ϱ, u, m, h and the triangulation $\mathcal{T}_h$.

3. The final estimate

In this section we use the assumption (34) (in order to obtain the optimum rate of convergence from Theorem 15, where the quantity m/ϱ appears (see Fig. 3)).

The preceding results (i.e., interpolation theorem, numerical integration and error of approximation of the boundary) yield then the following theorem:

16. Theorem. *Let $u \in H^2(\Omega)$, $f \in W^{1,\infty}(\widetilde{\Omega})$. Let assumption (34) and the assumptions concerning the degree of precision of the quadrature formulas be satisfied. Then*

$$\|\widetilde{u} - u_h\|_{1,\Omega_h} \leq \frac{C}{\sqrt{\varrho}}h \tag{38}$$

where the constant C does not depend on ϱ, m, h and the triangulation $\mathcal{T}_h$.

If in addition condition (36) is satisfied then

$$\|\widetilde{u} - u_h\|_{1,\Omega_h} \leq Ch \tag{39}$$

where again the constant C does not depend on ϱ, m, h and the triangulation $\mathcal{T}_h$.

4. General convergence theorem

In this section we will assume that $u \in H^1(\Omega)$ only and we will prove the convergence under a stronger assumption than (34), namely

$$C_1 h^{2-\delta} \leq \frac{\varrho}{m} \leq C_2 h^{2-\delta}\,, \tag{40}$$

where

$$0 < \delta < 1 \tag{41}$$

is a given number which can be arbitrary small and C_1, C_2 are positive constants. The lack of reguraltiy of $u \in H^1(\Omega)$ is usually a consequence of the fact that the Dirichlet condition is prescribed only on a part of Γ_1 or Γ_2 (and the Neumann condition is considered on the rest of Γ_1 or Γ_2).

Thus, using the preceding results we obtain:

17. Theorem. *Let Assumptions 1 as well as the assumptions concerning the degrees of precision of quadrature formulas on a triangle and its side (see Theorems 13 and 14) be satisfied. Then*

$$\lim_{h\to 0} \|\widetilde{u} - u_h\|_{1,\Omega_h} = 0\,, \tag{42}$$

where u_h is the solution of Discrete Problem 6, $u \in H^1(\Omega)$ is the solution of Continuous Problem 2 and $\widetilde{u} = E(u) \in H^1(\widetilde{\Omega})$ is its extension in the sense of Lemma 8 with $k = 1$.

5. Numerical results

Input:

$u(x,y) = \ln(x^2+y^2)$ ...the exact solution

$R_1 = 1$ ...radius of the circle inside

$\varrho = \frac{1}{50}R_1 = 0.02$ ...width of the gap between the circles

Notation:

m, d ...parameters of the finite element method (see Fig. 3)

u_h ...the approximation solution

$\|\widetilde{u} - u_h\|_{1,\Omega_h}$ ...the computed norm

EOC ...the experimental order of convergence where $EOC = \log_2 \frac{\|\widetilde{u}-u_h\|_{1,\Omega_h}}{\|\widetilde{u}-u_{h/2}\|_{1,\Omega_h}}$

$m = 16$

d	$\|\widetilde{u} - u_h\|_{1,\Omega_h}$	EOC
24	0.0536843809	
48	0.0267053823	1.01049656
96	0.0133371998	1.00244928
192	0.0066700114	0.99988775

$m = 32$

d	$\|\widetilde{u} - u_h\|_{1,\Omega_h}$	EOC
24	0.0536839448	
48	0.0267044940	1.01053294
96	0.0133354151	1.00259446
192	0.0066664393	1.00046766
384	0.0033347361	0.99939204

The computed results can be written in following form

$$\|\widetilde{u} - u_h\|_{1,\Omega_h} \le Ch$$

where the experimental order of convergence (EOC) gives the power of h.

We can formulate the conclusion now. The theory presented is concerned to semiregular triangles which are in a way specific and the experiment validate the theory.

Список литературы

1. Ciarlet, P. G.: The Finite Element Method for Elliptic Problems. Amsterdam: North-Holland, 1978.
2. Feistauer, M., Ženíšek, A.: Finite element solution of nonlinear elliptic problems. Numer. Math. 50 (1987), 451–475.
3. Hoderová, J.: Semiregular simplexes in the finite element method. Dissertation. Inst. of Mathematics FME BUT, Brno (2005), 89 p.

4. Křížek, M.: On semiregular families of triangulations and linear interpolation. Appl. Math. 36 (1991), 223–232.
5. Kufner, A., John, O., Fučík, S.: Function Spaces. Akademia, Praha, 1977.
6. Melkes, F., Zlámal, M.: Numerical solution of nonlinear quasistacionary magnetic fields. Int. J. Numer. Meth. Engng. 19 (1983), 1053–1062.
7. Nečas, J.: Les Métodes Directes en Théorie des Equations Elliptiques. Academia-Masson, Prague-Paris, 1967.
8. Nečas, J.: Introduction to the Theory of Nonlinear Elliptic Equations. Teubner Texte Zur Mathematik, Band 52, Leipzig, 1983.
9. Oganesian, L. A., Rukhovec, L. A.: Variational-Difference Methods for the Solution of Elliptic Problems. Izd. Akad. Nauk ArSSSR, Jerevan, 1979. (In Russian)
10. Ženíšek, A.: Nonlinear Elliptic and Evolution Problems and Their Finite Element Approximations. Academic Press, London, 1990.
11. Ženíšek, A.: The maximum angle condition in the finite element method for monotone problems with applications in magnetostatics. Numer. Math. 71 (1995), 399–417.
12. Ženíšek, A.: Finite element variational crimes in the case of semiregular elements. Applications of Math. 41 (1996), 367–398.
13. Zlámal, M.: On the finite element method. Numer. Math. 12 (1968), 394–409.
14. Zlámalová, J.: Semiregular finite elements in solving some nonlinear problem. Appl. Math. 46 (2001), 53–77.

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