CONVERGENCE CONDITIONS OF THE TWO-PARAMETRIC SECANT TYPE METHOD FOR SOLVING NONLINEAR EQUATIONS TAKING INTO ACCOUNT ERRORS

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Abstract. *In this paper we analyze the stability of the two-parametric secant type method to errors calculations for solving nonlinear equations and estimate the total error.*

1. Introduction

We consider the equation

$$F(x) = 0, \tag{1}$$

where F is a nonlinear operator defined on a convex subset D of a Banach space X with values in a Banach space Y. Studying solving methods for the equation (1) does not always take into account all the errors that arise during equation solving with the help of numerical methods. These issues were researched by certain authors. The stability and the error perturbation of the Newton-Kantorovich method and its modification are investigated in [5]. The evaluation of the total error of the simple iteration method is obtained in the work [3]. The paper [4] studies conditions of convergence and evaluation of the total error of the two-step iterative-differential method. The stability analysis of the accelerated Newton method to calculation's errors is carried out in [8].

In this paper we investigate convergence conditions of the two-parametric secant type method with regard to the rounding errors. The two-parametric secant type method, proposed in [6], has the form

$$x_{k+1} = x_k - [F(u_k, v_k)]^{-1} F(x_k),\ k = 0,\ 1,\ 2, \ldots, \tag{2}$$

where $F(u_k, v_k)$ is divided difference of the first order of the operator F at the points u_k and v_k, $u_k = x_k + a_k(x_{k-1} - x_k)$, $v_k = x_k + b_k(x_{k-1} - x_k)$, $a_k \in [-1, 1]$, $b_k \in [0, 1]$. In the work [7] the semilocal convergence of the method (2) is examined.

Definition 1. Let F be a nonlinear operator defined on a subset D of a linear space X with values in a linear space Y and let $x,\ y$ be two points of D. A linear operator from X into Y, denoted as $F(x, y)$, which satisfies the condition

$$F(x, y)(x - y) = F(x) - F(y).$$

is called a divided difference of F at the points x and y.

2. Convergence conditions of the perturbed method

Let us assume that the divided difference $F(x, y)$ is calculated with an error, the operator F is calculated exactly. Let us consider the perturbed iterative process

$$x_{k+1} = x_k - [F(u_k, v_k) + \Gamma_k]^{-1} F(x_k),\ k = 0,\ 1,\ 2, \ldots. \quad (3)$$

Here $\{\Gamma_k\} \in L(X, Y)$ is a sequence of linear operators. For the iterative process (3) the following theorem is valid.

Theorem 1. *Let $x_{-1}, x_0 \in D$ be initial approximations, $S_0 = \{x \in D : \|x - x_0\| < R\}$. Assume that the following conditions hold*

1) $\|x_{-1} - x_0\| = \alpha$;

2) there exist $A_0^{-1} = [F(u_0, v_0)]^{-1}$ and $\left\|A_0^{-1}\right\| \leq \beta_0$;

3) $\|F(x_0)\| \leq \zeta_0$, $\eta_0 = \beta_0\zeta_0$;

4) $\|\Gamma_k\| \leq \mu\eta_k$, $\beta_0\mu\eta_0 < 1$, $k = 0,\ 1,\ 2,\ \ldots$, where $\{\eta_k\}$ is a numerical sequence;

5) divided differences of the first order of the operator F satisfy Lipschitz condition

$$\|F(x, y) - F(u, v)\| \leq L(\|x - u\| + \|y - v\|), \quad x,\ y,\ u,\ v \in D, \quad L > 0$$

Let us denote

$$m = \beta_0 L \max\left\{\frac{\eta_0}{1 - \beta_0\mu\eta_0} + (a + b)\alpha,\ (1 + a + b)\frac{\eta_0}{1 - \beta_0\mu\eta_0}\right\} + \beta_0\mu\eta_0,$$

suppose that $|a_k| \leq a$, $b_k \leq b$ and the equation

$$u\left(1 - \frac{m}{1 - \beta_0 L((2 + a + b)u + (a + b)\alpha) - \beta_0\mu u}\right) - \frac{\eta_0}{1 - \beta_0\mu\eta_0} = 0 \quad (4)$$

has at least one positive zero, let R be the minimum positive one.

If $\beta_0 L((2 + a + b)R + (a + b)\alpha) + \beta_0\mu R < 1$,

$$M = \frac{m}{1 - \beta_0 L((2 + a + b)R + (a + b)\alpha) - \beta_0\mu R} < 1$$

and $\bar{S}_0 \subset D$, then the sequence $\{x_k\}$, given by the iterative process (3) is well defined, remains in $\bar{S}_0$ and converges to a unique solution $x_* \in \bar{S}_0$ of the equation (1). Moreover, the following inequality holds

$$\|x_k - x_*\| < \frac{1}{h(1 - M)} M^{\Phi_k}, \quad (5)$$

where $h = \dfrac{\beta_0\left[L(1 + a + b) + \mu\right]}{1 - \beta_0 L((2 + a + b)R + (a + b)\alpha) - \beta_0\mu R}$, $\Phi_{-1} = 0,\ \Phi_0 = 1,$
$\Phi_k = \Phi_{k-1} + \Phi_{k-2},\ k = 1,\ 2,\ \ldots$

Proof. Let us denote $A_k = F(u_k, v_k)$. By (3), we have

$$x_1 = x_0 - [A_0 + \Gamma_0]^{-1} F(x_0) = x_0 - [A_0(I + A_0^{-1}\Gamma_0)]^{-1} F(x_0) = \\ = x_0 - [I + A_0^{-1}\Gamma_0]^{-1} A_0^{-1} F(x_0).$$

Since $\|[I + A_0^{-1}\Gamma_0]^{-1}\| \leq \dfrac{1}{1 - \|A_0^{-1}\|\|\Gamma_0\|}$, then, taking into account the theorem's conditions, we get

$$\|x_1 - x_0\| = \left\| -[I + A_0^{-1}\Gamma_0]^{-1} A_0^{-1} F(x_0) \right\| \leq \frac{\left\|A_0^{-1}\right\| \|F(x_0)\|}{1 - \|A_0^{-1}\|\|\Gamma_0\|} \leq \\ \leq \frac{\eta_0}{1 - \beta_0 \mu \eta_0} = \frac{1}{h}\left(h\frac{\eta_0}{1 - \beta_0 \mu \eta_0}\right)^{\Phi_0} < R.$$

So, $x_1 \in S_0$.

Using the condition 5) of the theorem, we obtain

$$\left\|I - A_0^{-1} A_1\right\| \leq \left\|A_0^{-1}\right\| \|A_0 - A_1\| \leq \beta_0 L \left(\|u_0 - u_1\| + \|v_0 - v_1\|\right).$$

Since

$$\|u_0 - u_k\| = \|x_0 + a_0(x_{-1} - x_0) - x_k - a_k(x_{k-1} - x_k)\| \leq \\ \leq \|x_0 - x_k\| + |a_0| \|x_{-1} - x_0\| + |a_k| \|x_{k-1} - x_k\|,$$

$$\|v_0 - v_k\| = \|x_0 + b_0(x_{-1} - x_0) - x_k - b_k(x_{k-1} - x_k)\| \leq \\ \leq \|x_0 - x_k\| + b_0 \|x_{-1} - x_0\| + b_k \|x_{k-1} - x_k\|$$

and $|a_k| \leq a$, $b_k \leq b$, then

$$\left\|I - A_0^{-1} A_1\right\| \leq \beta_0 L \left((2 + a + b)\|x_0 - x_1\| + (a + b)\|x_{-1} - x_0\|\right) \leq \\ \leq \beta_0 L \left[(2 + a + b)\frac{\eta_0}{1 - \beta_0 \mu \eta_0} + (a + b)\alpha\right] < \beta_0 L \Big[(2 + a + b) R + (a + b)\alpha\Big] < 1.$$

By the Banach lemma, A_1^{-1} exists and

$$\left\|A_1^{-1}\right\| < \frac{\beta_0}{1 - \beta_0 L\left((2 + a + b) R + (a + b)\alpha\right)}.$$

Let us denote:

$$M_{k-1} = \frac{\beta_0 \Big[L\|x_k - x_{k-1}\| + L(a + b)\|x_{k-1} - x_{k-2}\| + \mu \eta_{k-1}\Big]}{1 - \beta_0 L \Big[(2 + a + b) R + (a + b)\alpha\Big]}, \ k \geq 1,$$

$$C = \frac{1 - \beta_0 L \Big[(2 + a + b) R + (a + b)\alpha\Big]}{1 - \beta_0 L \Big[(2 + a + b) R + (a + b)\alpha\Big] - \beta_0 \mu R}, \ C_0 = \frac{1}{1 - \beta_0 \mu \eta_0}.$$

It can be easily seen that $C_0M_0 \le M$ and $CM_k \le M$, $k \ge 1$.

From the definition of the first divided difference and (3) we can obtain

$$F(x_1) = F(x_0) - F(x_0, x_1)(x_0 - x_1) = (A_0 + \Gamma_0 - F(x_0, x_1))(x_0 - x_1).$$

Taking into account the theorem's condition 4) and Lipschitz condition 5), we get

$$\|F(x_1)\| \le \Big[\|A_0 - F(x_0, x_1)\| + \|\Gamma_0\|\Big]\|x_1 - x_0\| \le$$
$$\le \Big[L(\|u_0 - x_0\| + \|v_0 - x_1\|) + \mu\eta_0\Big]\|x_1 - x_0\| \le$$
$$\le \Big[L((a+b)\|x_0 - x_{-1}\| + \|x_1 - x_0\|) + \mu\eta_0\Big]\|x_1 - x_0\|.$$

Thus,

$$\left\|A_1^{-1}\right\|\|F(x_1)\| < \frac{\beta_0(L(a+b)\|x_0 - x_{-1}\| + L\|x_1 - x_0\| + \mu\eta_0)}{1 - \beta_0 L((2+a+b)R + (a+b)\alpha)} \frac{\eta_0}{1 - \beta_0\mu\eta_0} =$$
$$= M_0C_0\eta_0 = \eta_1.$$

Let us show that $\eta_1 < \eta_0$. In fact,

$$\eta_1 \le \frac{\beta_0 L\left((a+b)\alpha + \dfrac{\eta_0}{1-\beta_0\mu\eta_0}\right) + \beta_0\mu\eta_0}{(1 - \beta_0 L((2+a+b)R + (a+b)\alpha))(1 - \beta_0\mu\eta_0)}\eta_0 \le$$
$$\le \frac{m\eta_0}{1 - \beta_0 L((2+a+b)R + (a+b)\alpha) - \beta_0\mu\eta_0} \le$$
$$\le \frac{m\eta_0}{1 - \beta_0 L((2+a+b)R + (a+b)\alpha) - \beta_0\mu R} = M\eta_0 < \eta_0.$$

Above this we have

$$\eta_1 = \frac{CM_0}{C}\frac{\eta_0}{1-\beta_0\mu\eta_0} < \frac{M}{C}\frac{\eta_0}{1-\beta_0\mu\eta_0} < \frac{1}{Ch}\left(h\frac{\eta_0}{1-\beta_0\mu\eta_0}\right)^{\Phi_1}.$$

Therefore x_2 is well defined and

$$\|x_2 - x_1\| \le \left\|-[I + A_1^{-1}\Gamma_1]^{-1}\right\|\left\|A_1^{-1}\right\|\|F(x_1)\| \le$$
$$\le \frac{\left\|A_1^{-1}\right\|\|F(x_1)\|}{1 - \|A_1^{-1}\|\|\Gamma_1\|} < C\eta_1 < \frac{1}{h}\left(h\frac{\eta_0}{1-\beta_0\mu\eta_0}\right)^{\Phi_1}.$$

In addition, $\|x_2 - x_1\| < M\dfrac{\eta_0}{1-\beta_0\mu\eta_0}$. Since R is a solution of the equation (4), then

$$\|x_2 - x_0\| \le \|x_2 - x_1\| + \|x_1 - x_0\| < (M+1)\frac{\eta_0}{1-\beta_0\mu\eta_0} < R$$

and $x_2 \in S_0$.

Let us suppose that the following conditions are valid for $i = \overline{2, k-1}$:

- linear operators A_i are invertible,
- $\left\|A_i^{-1}\right\| \left\|F(x_i)\right\| < \eta_i = M_{i-1} C \eta_{i-1} < \dfrac{1}{Ch}\Big(h\dfrac{\eta_0}{1-\beta_0\mu\eta_0}\Big)^{\Phi_i}$, $\eta_i < \eta_{i-1}$,
- $\left\|x_{i+1} - x_i\right\| < C\eta_i < \dfrac{1}{h}\Big(h\dfrac{\eta_0}{1-\beta_0\mu\eta_0}\Big)^{\Phi_i} \le \dfrac{1}{h}M^{\Phi_i}$ and $x_{i+1} \in S_0$.

Then, for $i = k$ we also obtain

$$\left\|I - A_0^{-1}A_k\right\| \le \left\|A_0^{-1}\right\| \left\|A_0 - A_k\right\| \le \beta_0 L \left(\left\|u_0 - u_k\right\| + \left\|v_0 - v_k\right\|\right) \le$$
$$\le \beta_0 L \left[(2+a+b)\frac{\eta_0}{1-\beta_0\mu\eta_0} + (a+b)\,\alpha\right] < \beta_0 L \left[(2+a+b)\,R + (a+b)\,\alpha\right] < 1$$

and $\left\|A_k^{-1}\right\| < \dfrac{\beta_0}{1-\beta_0 L\left((2+a+b)\,R + (a+b)\,\alpha\right)}$.

From the definition of the first divided difference and (3) we can obtain

$$F(x_k) = F(x_{k-1}) - F(x_{k-1}, x_k)(x_{k-1} - x_k) =$$
$$= \left(A_{k-1} + \Gamma_{k-1} - F(x_{k-1}, x_k)\right)(x_{k-1} - x_k).$$

Taking into account the condition 5) of the theorem, we receive the following

$$\left\|F(x_k)\right\| = \left\|\left(A_{k-1} + \Gamma_{k-1} - F(x_{k-1}, x_k)\right)(x_{k-1} - x_k)\right\| \le$$
$$\le \Big[\left\|A_{k-1} - F(x_{k-1}, x_k)\right\| + \left\|\Gamma_{k-1}\right\|\Big] \left\|x_k - x_{k-1}\right\| \le$$
$$\le \Big[L\left(\left\|u_{k-1} - x_{k-1}\right\| + \left\|v_{k-1} - x_k\right\|\right) + \mu\eta_{k-1}\Big] \left\|x_k - x_{k-1}\right\| \le$$
$$\le \Big[L\left\|x_k - x_{k-1}\right\| + L(a+b)\left\|x_{k-1} - x_{k-2}\right\| + \mu\eta_{k-1}\Big] \left\|x_k - x_{k-1}\right\|.$$

Then

$$\left\|A_k^{-1}\right\| \left\|F(x_k)\right\| < \frac{\beta_0\Big[L\left\|x_k - x_{k-1}\right\| + L(a+b)\left\|x_{k-1} - x_{k-2}\right\| + \mu\eta_{k-1}\Big] C\eta_{k-1}}{1-\beta_0 L\left((2+a+b)\,R + (a+b)\,\alpha\right)} =$$
$$= M_{k-1} C \eta_{k-1} = \eta_k < \frac{\beta_0\Big[L\frac{1}{h}M^{\Phi_{k-1}} + L(a+b)\frac{1}{h}M^{\Phi_{k-2}} + \mu\frac{1}{Ch}M^{\Phi_{k-1}}\Big]\frac{1}{h}M^{\Phi_{k-1}}}{1-\beta_0 L\left((2+a+b)\,R + (a+b)\,\alpha\right)} <$$
$$< \frac{1}{Ch}M^{\Phi_{k-1}+\Phi_{k-2}} = \frac{1}{Ch}M^{\Phi_k}.$$

Since $M_{k-1}C \le M < 1$, then $\eta_k < \eta_{k-1}$.

Thus,

$$\left\|x_{k+1} - x_k\right\| < C\eta_k < \frac{1}{h}M^{\Phi_k}.$$

Obviously that for $i = \overline{1,k}$ $\|x_{i+1} - x_i\| < M^i \dfrac{\eta_0}{1-\beta_0\mu\eta_0}$ is valid. Therefore, taking into account that R is the solution of the equation (4), we get

$$\|x_{k+1} - x_0\| \le \|x_{k+1} - x_k\| + \|x_k - x_{k-1}\| + ... + \|x_1 - x_0\| <$$
$$< (M^k + M^{k-1} + ... + 1)\frac{\eta_0}{1-\beta_0\mu\eta_0} = \frac{1-M^{k+1}}{1-M}\frac{\eta_0}{1-\beta_0\mu\eta_0} < \frac{1}{1-M}\frac{\eta_0}{1-\beta_0\mu\eta_0} = R$$

and $x_{k+1} \in S_0$.

Let us show that $\{x_k\}$ is a Cauchy sequence. In fact,

$$\begin{aligned} &\|x_{k+p} - x_k\| \le \|x_{k+p} - x_{k+p-1}\| + ... + \|x_{k+1} - x_k\| < \\ &< (M^{p-1} + M^{p-2} + ... + 1)\frac{1}{h}\left(h\frac{\eta_0}{1-\beta_0\mu\eta_0}\right)^{\Phi_k} = \\ &= \frac{1-M^p}{h(1-M)}\left(h\frac{\eta_0}{1-\beta_0\mu\eta_0}\right)^{\Phi_k} < \frac{1}{h(1-M)}\left(h\frac{\eta_0}{1-\beta_0\mu\eta_0}\right)^{\Phi_k}. \end{aligned} \tag{6}$$

Therefore, $\{x_k\}$ is a Cauchy sequence and converges to $x_* \in \bar{S}_0$.

Now let us prove that x_* is a unique solution of the equation (1). Since

$$\|F(x_k)\| \le \left[L(1+a+b)\frac{\eta_0}{1-\beta_0\mu\eta_0} + \mu\eta_0\right]\|x_k - x_{k-1}\|$$

and $\|x_k - x_{k-1}\| \to 0$ if $k \to \infty$, then $F(x_*) = 0$.

Suppose that there exists $x_{**} \in \bar{S}_0$, $x_{**} \ne x_*$ i $F(x_{**}) = 0$. Let us denote $F(x_{**}, x_*) = H$. From the definition of divided difference of the first order we get

$$H(x_{**} - x_*) = F(x_{**}) - F(x_*).$$

If the operator H is invertible, then $x_{**} = x_*$. Indeed,

$$\left\|A_0^{-1}H - I\right\| = \left\|A_0^{-1}(H - A_0)\right\| \le \left\|A_0^{-1}\right\|\|H - A_0\| \le \beta_0 L\left[\|x_{**} - u_0\| + \|x_* - v_0\|\right] \le$$
$$\le \beta_0 L\left[\|x_{**} - x_0\| + \|x_* - x_0\| + (a+b)\|x_{-1} - x_0\|\right] < \beta_0 L(2R + (a+b)\alpha) < 1.$$

So, the operator H^{-1} exists. From (6) we can obtain the following estimation

$$\|x_k - x_*\| < \frac{1}{h(1-M)}M^{\Phi_k}.$$

□

3. Evaluation of the total error of the method (2)

Let us assume that the operator F is calculated approximately, i.e. we have perturbed equation

$$F_\varepsilon(x) = 0. \tag{7}$$

We assume that the operator F_ε is "close" to operator F in the sense that the following condition is valid

$$\|F_\varepsilon(x) - F(x)\| \le \delta(\varepsilon, x), \tag{8}$$

where $\delta(\varepsilon, x) \to 0$, if $\varepsilon \to 0$, $x \in D$.

Let us apply the method (2) for solving the equation (7)

$$x_{k+1}^\varepsilon = x_k^\varepsilon - [F_\varepsilon(u_k^\varepsilon, v_k^\varepsilon)]^{-1} F_\varepsilon(x_k^\varepsilon),\ k = 0,\ 1,\ 2, \dots . \tag{9}$$

For the iterative process (9) the following theorem is true.

Theorem 2. *Let us suppose*

1) conditions of Theorem 2 hold for the operator F_ε;

2) the equation (1) has at least one solution;

3) $\|[F(x,y)]^{-1}\| \le \beta$, for all $x, y \in D$;

4) the condition (8) holds.

If $k \to \infty$ and $\varepsilon \to 0$ then iterative process (9) converges to the solution $x_* \in \bar{S}_0$ of the equation (1) and the following inequality holds

$$\|x_k - x_*\| < \frac{1}{h(1-M)} M^{\Phi_k} + \beta\delta(\varepsilon, x). \tag{10}$$

Let us suppose that the divided difference $F(x, y)$ and F are calculated with errors. Then the following iterative process is studied

$$x_{k+1} = x_k - [F(u_k, v_k) + \Gamma_k]^{-1}[F(x_k) + \Psi_k],\ k = 0,\ 1,\ 2, \dots, \tag{11}$$

where $\{\Gamma_k\} \in L(X, Y)$ is a sequence of linear operators, $\{\Psi_k\} : X \to Y$ is a sequence of operators.

Theorem 3. *Let $x_{-1}, x_0 \in D$ be initial approximations, $S_0 = \{x \in D : \|x - x_0\| < R\}$. We assume that the following conditions hold*

1) $\|x_{-1} - x_0\| = \alpha$;

2) there exists $A_0^{-1} = [F(u_0, v_0)]^{-1}$ and $\left\|A_0^{-1}\right\| \le \beta_0$;

3) $\|F(x_0)\| \le \zeta_0$, $\eta_0 = \beta_0\zeta_0$;

4) $\|\Gamma_k\| \le \mu\eta_k$, $\beta_0\mu\eta_0 < 1$;

5) $\|\Psi_0\| \le \gamma\eta_0^2$, $\|\Psi_k\| \le \gamma\eta_k\eta_{k-1}$, $k \ge 1$;

6) divided differences of the first order of the operator F satisfy Lipschitz condition

$$\|F(x,y) - F(u,v)\| \le L(\|x-u\| + \|y-v\|),$$

where $x,\ y,\ u,\ v \in D,\ L > 0$.

Let us denote

$$m = \beta_0 L \max\left\{\frac{\eta_0(1+\beta_0\gamma\eta_0)}{1-\beta_0\mu\eta_0} + (a+b)\,\alpha, (1+a+b)\,\frac{\eta_0(1+\beta_0\gamma\eta_0)}{1-\beta_0\mu\eta_0}\right\} + \beta_0\mu\eta_0 + \beta_0\gamma\eta_0,$$

$$C^* = \frac{\beta_0\gamma\eta_0}{1-\beta_0 L\left((2+a+b)\,R + (a+b)\,\alpha\right)},$$

suppose that $|a_k| \le a,\ b_k \le b$ and

$$u\left(1 - \frac{m(1+C^*)}{1-\beta_0 L\left((2+a+b)\,u + (a+b)\,\alpha\right) - \beta_0\mu u}\right) - \frac{\eta_0(1+\beta_0\gamma\eta_0)}{1-\beta_0\mu\eta_0} = 0 \qquad (12)$$

has at least one positive zero, let R be the minimum positive one.

If $\beta_0 L\left((2+a+b)\,R + (a+b)\,\alpha\right) + \beta_0\mu R < 1$,

$$M = \frac{m}{1-\beta_0 L\left((2+a+b)\,R + (a+b)\,\alpha\right) - \beta_0\mu R}(1+C^*) < 1$$

and $\bar{S}_0 \subset D$, then the sequence $\{x_k\}$, given by iterative process (11), is well defined, remains in $\bar{S}_0$ and converges to a unique solution $x_* \in \bar{S}_0$ of the equation (1). Moreover, the following inequality holds

$$\|x_k - x_*\| < \frac{1}{h(1-M)} M^{\Phi_k},$$

where $h = \dfrac{\beta_0\Big(L(1+a+b) + \mu + \gamma\Big)(1+C^*)}{1-\beta_0 L\left((2+a+b)\,R + (a+b)\,\alpha\right) - \beta_0\mu R},\ \Phi_{-1} = 0,\ \Phi_0 = 1,$
$\Phi_k = \Phi_{k-1} + \Phi_{k-2},\ k = 1, 2, \ldots.$

Conclusion

In this paper we analyze the stability of the two-parametric secant type method to errors calculations for solving nonlinear equations and estimate the total error. It indicates that the iterative process (2) is resistant to the rounding errors and does not change the convergence order if conditions of Theorems are true.

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