A SIMULATION OF SUB-GAUSSIAN RANDOM FIELDS ON A SPHERE OF ORLICZ SPACES

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Abstract. *Estimates for the convergence speed models isotropic random fields on the sphere in the norms of Orlich space. The resulting estimates are used to construct models of random fields on the sphere. Models approximate the random field with given accuracy and reliability.*

Introduction

This paper continues investigation of convergence rate of the random series [3]–[7]. We obtain estimates for sub-Gaussian trigonometric series in Orlicz spaces. Same estimations of Gaussian series were obtained at [3]–[5], and on the uniform metric [6]. The results are used to model homogeneous and isotropic random fields on the sphere. Methods for the random modeling fields can be found in [2].

1. Basic determinations

Let (Ω, A, P) — be a standard probability space.

Definition 1. A random variable ξ is sub-Gaussian, if $E\xi = 0$ and $a \geq 0$ exists, such that for every $\lambda \in R^1$ following estimate occurs

$$E \exp\{\lambda\xi\} \leq \exp\left\{\frac{\lambda^2 a^2}{2}\right\}.$$

A space of sub-Gaussian variables $Sub(\Omega)$ is Banach relative to the following norm

$$\tau(\xi) = \sup_{\lambda \neq 0} \left[\frac{2 \ln E \exp\{\lambda\xi\}}{\lambda^2}\right]^{\frac{1}{2}}.$$

Definition 2. A family of random variables $S_\Lambda \subset Sub(\Omega)$ called strictly sub-Gaussian, if every finite or countable set of random variables $\{\xi_i, i \in I\} \subset S_\Lambda$ for every $\lambda \in R^1$ performs

$$\tau^2\left(\sum_{i \in I} \lambda_i \xi_i\right) = E\left(\sum_{i \in I} \lambda_i \xi_i\right).$$

Let $(T, \sum, \mu), \mu(T) < \infty$ — be some measurable space, $L_U(T)$ — Orlicz space, that was generated from C-function $U = \{U(x), x \in R^1\}$.

Definition 3. Orlicz space, generated by $U(x)$, called a function family $\{f(t), t \in T\}$, and for each function $f(t)$ exists constant r, that $\int_T U\left(\frac{f(t)}{r}\right) d\mu(t) < \infty$.

Space $L_U(T)$ is Banach relative to norm $\|f\|_{L_U} = \inf\left\{r > 0 : \int_T U\left(\frac{f(t)}{r}\right) d\mu(t) \leq 1\right\}$. Norm $\|f\|_{L_U}$ called the Luxemburg norm.

Definition 4. Let $f = \{f_k(t), t \in T, k = 1, 2...\}$ — be a family of functions from the space $L_U(T)$. This family belongs to the class $D_U(c)$, if numeric sequence $c = \{c_k, k = 1, 2, ...\}, 0 \leq c_k \leq c_{k+1}$ exists, such that for every sequence $r = \{r_k, k = 1, 2, ...\}$ following inequality holds

$$\left\| \sum_{k=1}^{n} r_k f_k(t) \right\|_{L_U} \leq c_n \left\| \sum_{k=1}^{n} r_k f_k(t) \right\|_{L_2}.$$

Definition 5. Isotropic in the broad sense field will be called linear isotropic field, if the random variables ξ_m^l are independent.

2. Simulation random fields on the sphere

Let S_d sphere in d — be a measurable space. A random continuous in mean-square homogeneous and isotropic field on the sphere $\xi(x)$ can be represented as [9]

$$\xi(x) = \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,d)} \xi_m^l S_m^l(x),$$

where ξ_m^l independent strictly sub-Gaussian random variables, $E\xi_m^l = 0$, $E\xi_m^l \xi_r^s = \sigma_m^2 \delta_m^r \delta_l^s$, $m = 0, 1, ...,$ $l = 1, ..., h(m, d)$, $S_m^l(x)$ — Spherical harmonic of m degree, $h(m, d)$ — harmonic count and $\sum_{m=0}^{\infty} \sigma_m^2 h(m, d) < \infty$.

Field model construct as

$$\xi_M(x) = \sum_{m=0}^{M} \sum_{l=1}^{h(m,d)} \xi_m^l S_m^l(x),$$

Number of summand M chosen in such way, where $\delta > 0$ and $0 < \alpha < 1$ and inequality holds $P\{\| \xi(x) - \xi_M(x) \| \geq \delta\} < 1 - \alpha$.

Next results were proved in papers [4, 5].

Lemma 1. *Let $\xi_1, \xi_2, ..., \xi_n$ — be an independent strictly sub-gaussian random variables, $E\xi_i^2 = \sigma_i^2, i = 1, 2, ..., n$. Then, for each $0 \leq u < 1$ and $N = 1, 2, ...$ following inequality* *holds*

$$E\exp\left\{\frac{u}{2Z_N}\sum_{l=1}^{n}\xi_l^2\right\}\leq \exp\left\{\frac{1}{2}\sum_{l=1}^{n}\frac{1}{l}\left(\frac{uZ_l}{Z_N}\right)^l\right\},$$

where $Z_N = \left(\sum_{i=1}^{n}\sigma_i^{2N}\right)^{\frac{1}{N}}$.

Lemma 2. *Let* $\xi_1, \xi_2, ..., \xi_n, ...$ *— independent strictly sub-gaussian random variables. If* $\sum_{i=1}^{\infty}\sigma_i^2 < \infty$, *then for each* $0 \leq u < 1$ *and* $N = 1, 2, ...$ *and following inequality holds*

$$E\exp\left\{\frac{u}{2Z_N}\sum_{i=1}^{\infty}\xi_i^2\right\}\leq \exp\left\{\frac{1}{2}\sum_{l=l}^{\infty}\frac{1}{l}\left(\frac{uZ_l}{Z_N}\right)^l\right\},$$

where $Z_N = \left(\sum_{i=1}^{\infty}\sigma_i^{2N}\right)^{\frac{1}{N}}$.

Lemma 3. *Let* $\xi_1, \xi_2, ..., \xi_n, ...$ *— be an independent strictly sub-Gaussian random variables. If* $\sum_{i=1}^{\infty}\sigma_i^2 < \infty$, *then for such* $0 \leq u < 1$ *and* $N = 1, 2, ...$ *following inequality holds*

$$E\exp\left\{\frac{u}{2Z_N}\sum_{i=1}^{\infty}\xi_i^2\right\}\leq \exp\left\{\frac{1}{2}v_N(u) + w_N(u)\right\},$$

where

$$w_N(u) = \frac{1}{2}\sum_{l=N}^{\infty}\frac{u^l}{l},$$

$$v_1(u) = 0, v_N(u) = \sum_{l=1}^{N-1}\frac{(lZ_l)^l}{lZ_N^l}.$$

Have similar lemma

Lemma 4. *If* $\left(\sum_{i=m}^{\infty}h(i,d)\sigma_i^{2N}\right)^{\frac{1}{N}} < \infty$, *for* $N = 1, 2, ...$ *then for each* $0 \leq u < 1$ *and* $m \geq 1$ *following inequality holds*

$$E\exp\left\{\frac{u}{2J(N,m)}\parallel \xi_m(x)\parallel_{L_2}^2\right\}\leq \exp\left\{\frac{1}{2}v_N(u) + w_N(u)\right\},$$

where $J(N,m) = \left(\sum_{i=m}^{\infty}h(i,d)\sigma_i^{2N}\right)^{\frac{1}{N}}$.

Using these results we obtain the following theorem.

Theorem 1. *If* $\left(\sum_{i=1}^{\infty} h(i,d)\sigma_i^{2N} \right)^{\frac{1}{N}} < \infty$, *for* $N = 1, 2, ...$ *then for each* $0 \le u < 1$ *and* $\varepsilon > 0$ *following inequality holds*

$$P\Big\{ \parallel \xi(x) - \xi_M(x) \parallel_{L_2} > \varepsilon \Big\} \le \exp\Big\{ -\frac{u\varepsilon^2}{2J(N, M+1)} \Big\} \exp\Big\{ \frac{1}{2} v_N(u) + w_N(u) \Big\},$$

where $w_N(u)$ *and* $v_N(u)$ *defined in Lemma 3,* $J(N,m)$ *— defined in Lemma 4.*

Proof. Compute $\parallel \xi(x) - \xi_M(x) \parallel_{L_2}^2 = \sum_{m=M+1}^{\infty} \sum_{l=1}^{h(m,d)} (\xi_m^l)^2$. According to Lemma 4 for $0 \le u < 1$ holds

$$E \exp\Big\{ \frac{u}{2J(N,M)} \parallel \xi(x) - \xi_M(x) \parallel_{L_2}^2 \Big\} \le \exp\Big\{ \frac{1}{2} v_N(u) + w_N(u) \Big\},$$

where $J(N,M) = \left(\sum_{i=M+1}^{\infty} h(i,d)\sigma_i^{2N} \right)^{\frac{1}{N}}$ and $N = 1, 2, ...$

Then, according to the Chebyshev inequality

$$P\Big\{ \parallel \xi(x) - \xi_M(x) \parallel_{L_2} > \varepsilon \Big\} = P\Big\{ \parallel \xi(x) - \xi_M(x) \parallel_{L_2}^2 > \varepsilon^2 \Big\} \le$$

$$\le E \exp\Big\{ \frac{u}{2J(N,M)} \parallel \xi(x) - \xi_M(x) \parallel_{L_2}^2 \Big\} \exp\Big\{ -\frac{u\varepsilon^2}{2J(N,M)} \Big\}.$$

Theorem proved. □

When $N = 1$ we have

$$P\Big\{ \parallel \xi(x) - \xi_M(x) \parallel_{L_2} > \varepsilon \Big\} \le \frac{\varepsilon}{(J(M))^{\frac{1}{2}}} \exp\Big\{ -\frac{\varepsilon^2}{2J(M)} \Big\} \exp\Big\{ \frac{1}{2} \Big\},$$

where $J(M) = \left(\sum_{i=M+1}^{\infty} h(i,d)\sigma_i^2 \right)$. . When $N = 2$ we have

$$P\Big\{ \parallel \xi(x) - \xi_M(x) \parallel_{L_2} > \varepsilon \Big\} \le \left(\frac{\varepsilon^2 - J(M)}{(J(2,M))} + 1 \right)^{\frac{1}{2}} \exp\Big\{ -\frac{\varepsilon^2 - J(M)}{2J(2,M)} \Big\},$$

where $J(2,M) = \left(\sum_{i=M+1}^{\infty} h(i,d)\sigma_i^4 \right)$.

Let

$$P_m(x) = \sum_{l=1}^{h(m,d)} \xi_m^l S_m^l(x),$$

$$Q_m^r(x) = \sum_{s=m}^{r} P_s(x),$$

$$R_m^r(x,b) = \sum_{s=m}^{r} b_s P_s(x),$$

where $\{b_s > 0\}$ — be a monotonically non-decreasing sequence. $R_m^r(x)$ — that trigonometric polynomial of $(d-1)$ — variable of order $m = (m, m, ..., m)$. that's why for $p > 2$ holds (Nikolskii inequality [8])

$$\| R_m^r(x,b) \|_{L_p} \leq 3^{d-1}(r)^{(d-1)\left(\frac{1}{2}-\frac{1}{p}\right)} \| R_m^r(x,b) \|_{L_2} .$$

Theorem 2. *Let a monotonically non-decreasing sequence exists $\{b_k > 0\}$, $b_k \to \infty$, $k \to \infty$, that following series convergent*

$$\sum_{s=1}^{\infty} c_s (J(s))^{\frac{1}{2}} < \infty$$

where

$$J(s) = \sum_{k=1}^{s} h(k,d) b_k^2 \sigma_k^2$$

and

$$c_s = 3^{d-1}(s)^{(d-1)\left(\frac{1}{2}-\frac{1}{p}\right)} \left(\frac{1}{b_s} - \frac{1}{b_{s+1}}\right),$$

Then, for each

$$\varepsilon > \sum_{s=M+1}^{\infty} c_s (J(s))^{\frac{1}{2}}$$

estimate holds

$$P\bigg\{ \| \xi(x) - \xi_M(x) \|_{L_p} > \varepsilon \bigg\} \leq \frac{\varepsilon}{(D(M))} \exp\bigg\{ -\frac{\varepsilon^2}{2D(M)^2} \bigg\} \exp\bigg\{ \frac{1}{2} \bigg\},$$

where $D(M) = \sum_{s=M+1}^{\infty} c_s (J(s))^{\frac{1}{2}}$.

Proof. Write Abel's transformation

$$Q_m^r(x) = \sum_{i=m}^{r-1} R_m^i(x,b) \left(\frac{1}{b_i} - \frac{1}{b_{i+1}}\right) + R_m^r(x,b) \frac{1}{b_{r+1}}.$$

Then

$$\| Q_m^r(x) \|_{L_p} = \sum_{i=m}^{r-1} \| R_m^i(x,b) \|_{L_p} \left(\frac{1}{b_i} - \frac{1}{b_{i+1}}\right) + \| R_m^r(x,b) \| + p \frac{1}{b_{r+1}} \leq$$

$$\sum_{i=m}^{r} c_i \| R_m^i(x,b) \|_{L_2},$$

where $c_i = 3^{d-1}(i)^{(d-1)\left(\frac{1}{2}-\frac{1}{p}\right)}\left(\frac{1}{b_i}-\frac{1}{b_{i+1}}\right)$, by $m \le i < r$, and $c_r = 3^{d-1}(r)^{(d-1)\left(\frac{1}{2}-\frac{1}{p}\right)}\left(\frac{1}{b_r}\right)$, by $i = r$.

Therefore, for some $y > 0$ holds

$$E\exp\left\{y^2 \parallel Q_m^r(x) \parallel_{L_p}^2\right\} \le E\exp\left\{\left(y\sum_{i=m}^{r} c_i \parallel R_m^i(x,b) \parallel_{L_2}\right)^2\right\}.$$

According to Jensen's inequality $\delta_i, i = m,..,r$ such that $\sum_{i=m}^{r}\delta_i = 1$, holds

$$E\exp\left\{\left(\sum_{i=m}^{r}\frac{y}{\delta_i}\delta_i c_i \parallel R_m^i(x,b) \parallel_{L_2}\right)^2\right\} \le E\exp\left\{\sum_{i=m}^{r}\delta_i\left(\frac{y}{\delta_i}c_i \parallel R_m^i(x,b) \parallel_{L_2}\right)^2\right\}.$$

According to the Holder inequality

$$E\exp\left\{y^2 \parallel Q_m^r(x) \parallel_{L_p}^2\right\} \le \prod_{i=m}^{r}\left(E\exp\left\{\left(\frac{y}{\delta_i}c_i \parallel R_m^i(x,b) \parallel_{L_2}\right)^2\right\}\right)^{\delta_i}.$$

Mark $u_i = 2y^2c_i^2\delta_i^{-2}J(N,m,i)$, where

$$J(N,m,i) = \left(\sum_{k=m}^{i} h(k,d)\sigma_k^{2N}b_k^{2N}\right)^{\frac{1}{N}}.$$

If $0 \le u_i = 2y^2c_i^2\delta_i^{-1}J(N,m,i) < 1$, then by Lemma 4

$$E\exp\left\{\left(\frac{y}{\delta_i}c_i \parallel R_m^i(x,b) \parallel_{L_2}\right)^2\right\} =$$

$$E\exp\left\{\frac{2y^2c_i^2\delta_i^{-2}J(N,m,i)}{2J(N,m,i)}\left(\parallel R_m^i(x,b) \parallel_{L_2}\right)^2\right\} \le$$

$$\exp\left\{\frac{1}{2}v_N(u_i) + w_N(u_i)\right\}.$$

If $N = 1$, then $E\exp\left\{\left(\frac{y}{\delta_i}c_i \parallel R_m^i(x,b) \parallel_{L_2}\right)^2\right\} = (1-u_i)^{\frac{1}{2}}$.

Then

$$E\exp\left\{y^2 \parallel Q_m^r(x) \parallel_{L_p}^2\right\} \le \prod_{i=m}^{r}\left((1-u_i)^{-\frac{1}{2}}\right)^{\delta_i} = \exp\left\{-\frac{1}{2}\sum_{i=m}^{r}\delta_i\ln(1-u_i)\right\}.$$

Set $\delta_i = \frac{\sqrt{2}yc_iJ^{\frac{1}{2}}(m,i)}{V}$, where $V > 0$ and $\sum_{i=m}^{r}\delta_i = 1$.

Then

$$\sum_{i=m}^{r}\frac{\sqrt{2}yc_iJ^{\frac{1}{2}}(m,i)}{V} = \frac{\sqrt{2}y}{V}\sum_{i=m}^{r}c_iJ^{\frac{1}{2}}(m,i) = 1,$$

or

$$V=\sqrt{2}y\sum_{i=m}^{r}c_iJ^{\frac{1}{2}}(m,i).$$

And therefore,

$$u_i=2y^2c_i^2\delta_i^{-2}J(N,m,i)=2y^2\left(\sum_{i=m}^{r}c_iJ^{\frac{1}{2}}(m,i)\right)^2=V^2.$$

If $V<1$, then

$$E\exp\{y^2\parallel Q_m^r(x)\parallel_{L_p}^2\}\leq\exp\{-\frac{1}{2}\sum_{i=m}^{r}\delta_i\ln(1-u_i)\}=$$

$$\exp\left\{-\frac{1}{2}\ln(1-V^2)\sum_{i=m}^{r}\frac{\sqrt{2}yc_iJ^{\frac{1}{2}}(m,i)}{V}\right\}=(1-V^2)^{-\frac{1}{2}}.$$

Let set $y^2=\frac{V^2}{2}\left(\sum_{i=m}^{r}c_iJ^{\frac{1}{2}}(m,i),\right)^{-2}$, then

$$E\exp\left\{\frac{V^2}{2\left(\sum_{i=m}^{r}c_iJ^{\frac{1}{2}}(m,i),\right)^2}\parallel Q_m^r(x)\parallel_{L_p}^2\right\}\leq(1-V^2)^{-\frac{1}{2}}.$$

Consequently, according to the Chebyshev inequality,

$$P\{\parallel Q_m^r(x)\parallel_{L_p}^2>\varepsilon^2\}\leq\exp\left\{-\frac{V^2\varepsilon^2}{2\left(\sum_{i=m}^{r}c_iJ^{\frac{1}{2}}(m,i),\right)^2}\right\}(1-V^2)^{-\frac{1}{2}}.$$

If the series of converges $\sum_{i=1}^{\infty}c_iJ^{\frac{1}{2}}(1,i)$, then $\sum_{i=m}^{r}c_iJ^{\frac{1}{2}}(m,i)\to 0$ where $m\to\infty$, $r\to\infty$.

Consequently, $P\{\parallel Q_m^r(x)\parallel_{L_p}^2>\varepsilon^2\}\to 0$ where $m\to\infty, r\to\infty$. If we set $m=M+1$ and direct $r\to\infty$, then we will get following estimate

$$P\{\parallel Q_M^\infty(x)\parallel_{L_p}^2>\varepsilon^2\}\leq\exp\left\{-\frac{V^2\varepsilon^2}{2\left(\sum_{i=M+1}^{\infty}c_iJ^{\frac{1}{2}}(M+1,i)\right)^2}\right\}(1-V^2)^{-\frac{1}{2}}.$$

If we optimize right part by V, i.e., when

$$\varepsilon>\sum_{i=M+1}^{\infty}c_iJ^{\frac{1}{2}}(M+1,i)$$

set $V=1-\frac{1}{\varepsilon}\sum_{i=M+1}^{\infty}c_iJ^{\frac{1}{2}}(M+1,i)$, then we get estimate. Theorem proved. □

With a similar argument we can prove a next theorem.

Theorem 3. *If sequence convergence*

$$\sum_{i=1}^{\infty} C_i \frac{h(i,d)\sigma_i^2}{\left(\sum_{k=i}^{\infty} h(k,d)\sigma_k^2\right)^{\frac{1}{2}}} < \infty,$$

where $C_i = 3^{d-1}(i)^{(d-1)\left(\frac{1}{2}-\frac{1}{p}\right)}$, *then for each* $\varepsilon > G(M+1)$ *holds next estimate*

$$P\left\{ \parallel \xi(x) - \xi_M(x) \parallel_{L_p} > \varepsilon \right\} \leq$$

$$\frac{\varepsilon}{(G(M+1))} \exp\left\{ -\frac{\varepsilon^2}{2G^2(M+1)} \right\} \exp\left\{ \frac{1}{2} \right\},$$

where

$$G^2(M+1) = (1+\sqrt{2}) \sum_{i=M+1}^{\infty} C_i \frac{h(i,d)\sigma_i^2}{\left(\sum_{k=i}^{\infty} h(k,d)\sigma_k^2\right)^{\frac{1}{2}}}.$$

Proof. For chosen sequence $\{b_k\}$ get

$$J^{\frac{1}{2}}(M+1,i) = \left(\sum_{k=M+1}^{i} h(k,d)\sigma_k^2 b_k^2 \right)^{\frac{1}{2}} +$$

$$1+\sqrt{2}\left(\left(\sum_{k=i}^{\infty} h(k,d)\sigma_k^2\right)^{-1} - 1 \right)^{\frac{1}{2}}$$

and

$$\sum_{i=M+1}^{\infty} C_i J^{\frac{1}{2}}(M+1,i) \leq \sum_{i=M+1}^{\infty} C_i \left(\frac{1}{b_i} - \frac{1}{b_{i+1}}\right) \left(1+\sqrt{2}\left(\left(\sum_{k=i}^{\infty} h(k,d)\sigma_k^2\right)^{-1} - 1 \right)^{\frac{1}{2}} \right) \leq$$

$$\left(1+\sqrt{2}\right) \sum_{i=M+1}^{\infty} C_i \frac{h(i,d)\sigma_i^2}{\left(\sum_{k=i}^{\infty} h(k,d)\sigma_k^2\right)^{\frac{1}{2}}}.$$

Theorem proved. □

In modeling of random fields ask the modeling accuracy $\varepsilon > 0$ and reliability $1-\alpha$, $0 < \alpha < 1$. For space L_2 number of summand M in model (1) we found as minimum value, where inequality when $N = 1$

$$\frac{\varepsilon}{(J(M))^{\frac{1}{2}}} \exp\left\{ -\frac{\varepsilon^2}{2J(M)} \right\} \exp\left\{ \frac{1}{2} \right\} \leq 1-\alpha,$$

And when $N = 2$ inequality

$$\left(\frac{\varepsilon^2 - J(M)}{(J(2,M))} + 1 \right)^{\frac{1}{2}} \exp\left\{ -\frac{\varepsilon^2 - J(M)}{2J(2,M)} \right\} \leq 1-\alpha$$

For functional space $L_p, p > 2$ number of summand M in model (1) we found from inequality

$$\frac{\varepsilon}{(D(M+1))}\exp\Big\{-\frac{\varepsilon^2}{2D^2(M+1)}\Big\}\exp\Big\{\frac{1}{2}\Big\}\leq 1-\alpha.$$

Left-side depends on the sequence $\{b_k\}$. As $\sum_{i=1}^{\infty} h(i,d)\sigma_i^2 < \infty$, then, without any loss of generality, we can assume that $\sum_{i=1}^{\infty} h(i,d)\sigma_i^2 = 1$ and choose $b_k = 1+\Bigg(\big(\sum_{i=k}^{\infty} h(i,d)\sigma_i^2\big)^{-1}-1\Bigg)$.

Consequently, number of summand in model $\xi_M(x)$ we can calculate from inequality

$$\frac{\varepsilon}{(G(M+1))}\exp\Big\{-\frac{\varepsilon^2}{2G^2(M+1)}\Big\}\exp\Big\{\frac{1}{2}\Big\}\leq 1-\alpha.$$

With a similar argument we can prove a next theorem

Theorem 4. *Let $U(x) = \{U(x), x \in \mathbb{R}$ be a C-Orlicz function, those function*

$$GU(x) = \exp\big\{(U^{(-1)}(x-1))^2\big\}, x \geq 1$$

convex at $x \geq 1$, $U^{(-1)}(x)$ - inverse function to $U(x)$. Then for every x such $x \geq \max(\mu(T),1)\tau\big(2+(U^{(-1)}(1))^{-2}\big)^{\frac{1}{2}}$, following inequality holds

$$P\Big\{\parallel \xi(x)-\xi_M(x)\parallel_{L_U(x)} > \varepsilon\Big\}\leq$$

$$\frac{\varepsilon U^{(-1)}(1)}{\max(1,\mu(T))\tau}\exp\Big\{-\frac{\varepsilon^2(U^{(-1)}(1))^2}{2(\max(1,\mu(T))\tau)^2}\Big\}\exp\Big\{\frac{1}{2}\Big\},$$

Theorem 5. *Let $\xi(x)$ — be a strictly Orlicz field, $\xi_M(x)$ — those field model. If some $p > 2$ sequence convergence $\sum_{m=1}^{\infty} h(m,d)\sigma_m^2 m^{(d-1)\left(2-\frac{2}{p}\right)}$, then for any $\delta > 0$ following inequality holds*

$$P\Bigg\{\Bigg(\int\limits_{S_d}|\xi(x)-\xi_M(x)|^p dx\Bigg)^{\frac{1}{p}}\geq\delta\Bigg\}\leq\Bigg(U\Bigg(\delta^2\Bigg[C_{u_p}C(3^{d-1})^2\sum_{m=M+1}^{\infty}h(m,d)\sigma_m^2 m^{(d-1)\left(2-\frac{2}{p}\right)}\Bigg]^{-\frac{p}{2}}\Bigg)\Bigg)^{-1}.$$

Proof. Let use Nikolskii inequality. We have

$$P\Bigg\{\Bigg(\int\limits_{S_d}|\xi(x)-\xi_M(x)|^p dx\Bigg)^{\frac{1}{p}}\geq\delta\Bigg\}= P\Bigg\{\Bigg(\int\limits_{S_d}|\sum_{m=M+1}^{\infty}h(m,d)\sigma_m^2 m^{(d-1)\left(2-\frac{2}{p}\right)}|^p dx\Bigg)^{\frac{1}{p}}\geq\delta\Bigg\}$$

$$\leq\Bigg(U\Bigg(\delta^2\Bigg[C_{u_p}\sum_{m=M+1}^{\infty}\|\xi_m^l\|_{u_p}^2\big(\int\limits_T|S_m^l(x)|^p dx\big)^{\frac{2}{p}}\Bigg]^{-\frac{p}{2}}\Bigg)\Bigg)^{-1}.$$

As the $S_m^l(x)$ — a trigonometric polynomial of $(d-1)$ variables, then for $p > 2$ inequality holds $\|S_m^l\|_{L_p} \leq 3^{d-1}m^{(d-1)\left(2-\frac{2}{p}\right)}\|S_m^l\|_{L_2}$, a $\|S_m^l(x)\|_{L_2} = 1$. Therefore, when

$p > 2$ we have $\left(\int_T |S_m^l(x)|^p dx\right)^{\frac{1}{p}} \leq 3^{d-1} m^{(d-1)\left(2-\frac{2}{p}\right)}$ and $\|S_m^l\|_{u_p}^2 \leq C\sigma_m^2$. In that case following inequality holds

$$C_{u_p} \sum_{m=M+1}^{\infty} \sum_{l=1}^{h(m,d)} \|\xi_m^l\|_{u_p}^2 \Big(\int_T |S_m^l(x)|^p dx\Big)^{\frac{2}{p}} \leq C_{u_p} C (3^{d-1})^2 \sum_{m=M+1}^{\infty} h(m,d)\sigma_m^2 m^{(d-1)\left(2-\frac{2}{p}\right)}.$$

Theorem proved. □

When $p = 2$ holds following theorem

Theorem 6. *Let $\xi(x)$ - be a strictly Orlicz field, $\xi_M(x)$ - those field model. If such sequence convergence $\sum_{m=1}^{\infty} h(m,d)\sigma_m^2 m^{(d-1)}$, then for each $\delta > 0$ holds inequality*

$$P\left\{ \left(\int_{S_d} |\xi(x) - \xi_M(x)|^2 dx \right)^{\frac{1}{2}} \geq \delta \right\} \leq \left(U\left(\delta^2 \left[C_{u_p} C \sum_{m=M+1}^{\infty} h(m,d)\sigma_m^2 m^{(d-1)} \right] \right) \right)^{-1}.$$

Conclusion

The paper constructed a model of random fields on the sphere. The models of linear isotropic fields from Orlicz space were observed. The models approximate the field with given accuracy and reliability.

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