THE SCHEME OF PARTIAL AVERAGING FOR ONE CLASS OF HYBRID SYSTEMS

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Abstract. *This paper contains the substantiation of the scheme of partial averaging for one class of hybrid systems where one equation is a differential equation with Hukuhara derivative and the other one is an ordinary differential equation.*

Introduction

In practice there often appear the so-called hybrid systems — systems which contain equations of different nature: for example, one of the equations is a partial differential equation and the other one is an ordinary differential equation, or one of the equations is a discrete one and the other is a differential equation, etc. In this paper we consider the case of a hybrid system, when one of the equations is a differential equation with Hukuhara derivative and the other one is an ordinary differential equation. The interest in such systems follows from the fact, that some parameters of the model can be accurate, while the rest may contain the noise, errors and inaccuracies.

1. Main Definitions

Development of the theory of multivalued mappings led to the question what should be understood as a derivative of a multivalued mapping. The main cause of difficulties for the inducting of such definition was the nonlinearity of the space $comp(R^n)$, which led to the absence of the concept of difference. There are several approaches to define the difference of two sets, one of them is the Hukuhara difference.

Definition 1. [see [6]] Let $X, Y \in conv(R^n)$. The set $Z \in conv(R^n)$, where $X = Y + Z$, is called the Hukuhara difference of sets X and Y and is designated as $X \overset{h}{-} Y$.

Along with the inducted difference there appeared the concept of derivative.

Definition 2. [see [6]] A multuvalued mapping $X : I \to conv(R^n), I \subset R$, is called differentiable in the sense of Hukuhara at point $t \in I$ if there exists such $D_H X(t) \in conv(R^n)$ that the limits $\lim\limits_{\Delta t \to 0} \frac{1}{\Delta t} \left(X(t + \Delta t) \overset{h}{-} X(t) \right)$ and $\lim\limits_{\Delta t \to 0} \frac{1}{\Delta t} \left(X(t) \overset{h}{-} X(t - \Delta t) \right)$ exist and are equal to $D_H X(t)$. The set $D_H X(t)$ is called the Hukuhara derivative of the multivalued mapping $X : I \to conv(R^n)$ at point t.

In 1969 F.S. de Blasi and F. Iervolino first considered the differential equation with Hukuhara derivative [4, 2, 3, 1], which solution was a multivalued mapping. After that various existence, uniqueness theorems were proved, stability of solutions for this type of equations was investigated, integro-differential equations, impulse differential equations, differential equations with fractional derivatives, controlled differential equations with Hukuhara derivative were considered. The possibility of using some averaging schemes for such type of equations was studied in [5, 13, 11, 7, 12, 8, 9, 10].

Consider the hybrid system

$$\begin{cases} D_H X = F(t, X, y), \\ \dot{y} = g(t, X, y), \\ X(t_0) = X_0, \\ y(t_0) = y_0, \end{cases} \tag{1}$$

where $I = [t_0, T] \subset R$; $X : I \to conv(R^n)$ is a multivalued mapping; $y : I \to R^m$ is a vector function; $F : I \times conv(R^n) \times R^m \to conv(R^n)$ is a multivalued mapping; $g : I \times conv(R^n) \times R^m \to R^m$ is a vector function; $X_0 \in conv(R^n)$, $y_0 \in R^m$.

Consider a class S of pairs $(X(\cdot), y(\cdot))$, where $X(\cdot)-$ is a continuously differentiable on I in a sense of Hukuhara multivalued mapping, $y(\cdot)-$ is a continuously differential on I vector-function.

Definition 3. A pair $(X(\cdot), y(\cdot)) \in S$ is called a solution of system (1), if it satisfies the system for all $t \in I$ (e.g for all $t \in I$ the following equalities fulfill $D_H X(t) = F(t, X(t), y(t))$, $\dot{y}(t) = g(t, X(t), y(t)))$ and $X(t_0) = X_0$, $y(t_0) = y_0$.

Theorem 1. *Let in the domain*

$$Q = \{(t, X, y) : \ t_0 \le t \le t_0 + a, \ h(X, X_0) \le b, \ \|y - y_0\| \le c\}$$

the multivalued mapping $F(t, X, y)$ and the vector function $g(t, X, y)$ be continuous and satisfy the Lipschitz condition in variables X and y, i.e. there exists such constant $\lambda > 0$ that

$$h\left(F(t, X_1, y_1), F(t, X_2, y_2)\right) \le \lambda \left[h(X_1, X_2) + \|y_1 - y_2\|\right],$$
$$\|g(t, X_1, y_1) - g(t, X_2, y_2)\| \le \lambda \left[h(X_1, X_2) + \|y_1 - y_2\|\right].$$

Then system (1) has the unique solution defined on the interval $[t_0, t_0 + d]$ where $d = \min\left(a, \frac{b}{M}, \frac{c}{M}\right)$, constant M satisfies inequalities $|F(t, X, y)| \le M$, $\|g(t, X, y)\| \le M$ in the domain Q.

2. Main Results

Consider the hybrid system with a small parameter

$$\begin{cases} D_H X = \varepsilon F(t, X, y), \\ \dot{y} = \varepsilon g(t, X, y), \\ X(0) = X_0, \\ y(0) = y_0, \end{cases} \tag{2}$$

where $t \geq 0$ is time, $X \in D_1 \subset conv(R^n)$, $y \in D_2 \subset R^m$, $\varepsilon > 0$ is a small parameter.

With system (2) the following partially averaged system is assigned:

$$\begin{cases} D_H \bar{X} = \varepsilon \bar{F}(t, \bar{X}, \bar{y}), \\ \dot{\bar{y}} = \varepsilon \bar{g}(t, \bar{X}, \bar{y}), \\ \bar{X}(0) = X_0, \\ \bar{y}(0) = y_0, \end{cases} \tag{3}$$

where

$$\lim_{T\to\infty} \frac{1}{T} h \left(\int_0^T F(t, X, y)\, dt, \int_0^T \bar{F}(t, X, y)\, dt \right) = 0, \tag{4}$$

$$\lim_{T\to\infty} \frac{1}{T} \left\| \int_0^T g(t, X, y)\, dt - \int_0^T \bar{g}(t, X, y)\, dt \right\| = 0. \tag{5}$$

Theorem 2. *Let in the domain* $Q = \{(t, X, y) : \ t \geq 0, X \in D_1, y \in D_2\}$ *the following conditions hold:*

1) the multivalued mappings $F(t, X, y), \bar{F}(t, X, y)$ *and vector functions* $g(t, X, y), \bar{g}(t, X, y)$ *are continuous in* t*, uniformly bounded with constant* M *and satisfy the Lipschitz condition in* X *and* y *with constant* λ*, i.e.*

$$|F(t, X, y)| \leq M, h\left(F(t, X_1, y_1), F(t, X_2, y_2)\right) \leq \lambda \left[h(X_1, X_2) + \|y_1 - y_2\|\right],$$

$$\left|\bar{F}(t, X, y)\right| \leq M, h\left(\bar{F}(t, X_1, y_1), \bar{F}(t, X_2, y_2)\right) \leq \lambda \left[h(X_1, X_2) + \|y_1 - y_2\|\right],$$

$$\|g(t, X, y)\| \leq M, \|g(t, X_1, y_1) - g(t, X_2, y_2)\| \leq \lambda \left[h(X_1, X_2) + \|y_1 - y_2\|\right],$$

$$\|\bar{g}(t, X, y)\| \leq M, \|\bar{g}(t, X_1, y_1) - \bar{g}(t, X_2, y_2)\| \leq \lambda \left[h(X_1, X_2) + \|y_1 - y_2\|\right];$$

2) limits (4) and (5) exist uniformly with respect to $X \in D_1$ *and* $y \in D_2$*;*

3) the solution $\left(\bar{X}(t), \bar{y}(t)\right)$ *of system (3) with the initial condition* $\bar{X}(0) = X_0 \in D_1' \subset D_1$*,* $\bar{y}(0) = y_0 \in D_2' \subset D_2$ *is defined for all* $t \geq 0$*,* $\varepsilon \in (0, \sigma]$ *and* $\bar{X}(t)$ *belongs with some* ρ*- neighborhood to the domain* D_1*,* $\bar{y}(t)$ *belongs with some* ξ*-neighborhood to the domain* D_2*.*

Then for any $\eta > 0$ and $L > 0$ there exists such $\varepsilon_0(\eta, L) \in (0,\sigma]$ that for varepsilon $\in (0,\varepsilon_0]$ and $t \in [0, L\varepsilon^{-1}]$ the following inequalities fulfill:

$$h\left(X(t), \bar{X}(t)\right) < \eta, \|y(t) - \bar{y}(t)\| < \eta,$$

where $(X(\cdot), y(\cdot))$ and $\left(\bar{X}(\cdot), \bar{y}(\cdot)\right)$ are the solutions of systems (2) and (3) with the initial conditions $X(0) = \bar{X}(0) \in D_1'$, $y(0) = \bar{y}(0) \in D_2'$.

Proof. From conditions 1) and 2) of the theorem it follows that systems (2) and (3) have unique solutions that are defined for $t \geq 0$ if $X(t)$ and $y(t)$ (accordingly $\bar{X}(t)$ and $\bar{y}(t)$) belong to the domains D_1, D_2. That is why for $D_1 = conv(R^n)$, $D_2 = R^m$ condition 3) follows from 1) and 2).

Replace systems (2) and (3) with the equivalent system of integral equations:

$$\begin{cases} X(t) = X_0 + \varepsilon \int\limits_0^t F(s, X(s), y(s))\, ds, \\ y(t) = y_0 + \varepsilon \int\limits_0^t g(s, X(s), y(s))\, ds, \end{cases} \tag{6}$$

$$\begin{cases} \bar{X}(t) = X_0 + \varepsilon \int\limits_0^t \bar{F}(s, \bar{X}(s), \bar{y}(s))\, ds, \\ \bar{y}(t) = y_0 + \varepsilon \int\limits_0^t \bar{g}(s, \bar{X}(s), \bar{y}(s))\, ds. \end{cases} \tag{7}$$

Then

$$h\left(X(t), \bar{X}(t)\right) =$$

$$= h\left(X_0 + \varepsilon \int\limits_0^t F(s, X(s), y(s))ds, X_0 + \varepsilon \int\limits_0^t \bar{F}(s, \bar{X}(s), \bar{y}(s))ds\right) =$$

$$= h\left(\varepsilon \int\limits_0^t F(s, X(s), y(s))ds, \varepsilon \int\limits_0^t \bar{F}(s, \bar{X}(s), \bar{y}(s))ds\right) \leq$$

$$\leq h\left(\varepsilon \int\limits_0^t F(s, X(s), y(s))ds, \varepsilon \int\limits_0^t F(s, \bar{X}(s), \bar{y}(s))ds\right) +$$

$$+h\left(\varepsilon \int\limits_0^t F(s, \bar{X}(s), \bar{y}(s))ds, \varepsilon \int\limits_0^t \bar{F}(s, \bar{X}(s), \bar{y}(s))ds\right) \leq$$

$$\leq \varepsilon \int\limits_0^t h\left(F(s, X(s), y(s)), F(s, \bar{X}(s), \bar{y}(s))\right) ds+$$

$$+\varepsilon h\left(\int\limits_0^t F(s,\bar{X}(s),\bar{y}(s))ds, \int\limits_0^t \bar{F}(s,\bar{X}(s),\bar{y}(s))ds\right) \leq$$

$$\leq \varepsilon\lambda\int\limits_0^t \left[h(X(s),\bar{X}(s)) + \|y(s)-\bar{y}(s)\|\right]ds+$$

$$+\varepsilon h\left(\int\limits_0^t F(s,\bar{X}(s),\bar{y}(s))ds, \int\limits_0^t \bar{F}(s,\bar{X}(s),\bar{y}(s))ds\right). \tag{8}$$

Similarly

$$\|y(t)-\bar{y}(t)\| \leq$$

$$\leq \varepsilon\lambda\int\limits_0^t \left[h(X(s),\bar{X}(s)) + \|y(s)-\bar{y}(s)\|\right]ds+$$

$$+\varepsilon\left\|\int\limits_0^t \left[g(s,\bar{X}(s),\bar{y}(s)) - \bar{g}(s,\bar{X}(s),\bar{y}(s))\right]ds\right\|. \tag{9}$$

Divide the interval $[0, L\varepsilon^{-1}]$ in p equal intervals by the points $t_i = \frac{iL}{\varepsilon p}, i = \overline{0,p}$. Define by $(\bar{X}_i, \bar{y}_i) = (\bar{X}(t_i), \bar{y}(t_i))$ the solution of system (2) in division points.

Let us estimate the expressions $\varepsilon h\left(\int\limits_0^t F(s,\bar{X}(s),\bar{y}(s))ds, \int\limits_0^t \bar{F}(s,\bar{X}(s),\bar{y}(s))ds\right)$ and $\varepsilon\left\|\int\limits_0^t \left[g(s,\bar{X}(s),\bar{y}(s)) - \bar{g}(s,\bar{X}(s),\bar{y}(s))\right]ds\right\|$ in the interval $[t_k, t_{k+1}]$, where $0 \leq k \leq p-1$.

$$\varepsilon h\left(\int\limits_0^t F(s,\bar{X}(s),\bar{y}(s))ds, \int\limits_0^t \bar{F}(s,\bar{X}(s),\bar{y}(s))ds\right) =$$

$$= \varepsilon h\left(\sum_{i=0}^{k-1}\int\limits_{t_i}^{t_{i+1}} F(s,\bar{X}(s),\bar{y}(s))ds + \int\limits_{t_k}^{t} F(s,\bar{X}(s),\bar{y}(s))ds,\right.$$

$$\left.\sum_{i=0}^{k-1}\int\limits_{t_i}^{t_{i+1}} \bar{F}(s,\bar{X}(s),\bar{y}(s))ds + \int\limits_{t_k}^{t} \bar{F}(s,\bar{X}(s),\bar{y}(s))ds\right) \leq$$

$$\leq \varepsilon\left[\sum_{i=0}^{k-1} h\left(\int\limits_{t_i}^{t_{i+1}} F(s,\bar{X}(s),\bar{y}(s))ds, \int\limits_{t_i}^{t_{i+1}} \bar{F}(s,\bar{X}(s),\bar{y}(s))ds\right)+\right.$$

$$+h\left(\int\limits_{t_k}^{t} F(s,\bar X(s),\bar y(s))ds, \int\limits_{t_k}^{t} \bar F(s,\bar X(s),\bar y(s))ds\right)\Bigg] \le$$

$$\le \varepsilon\left[\sum_{i=0}^{k-1}\left(h\left(\int\limits_{t_i}^{t_{i+1}} F(s,\bar X(s),\bar y(s))ds, \int\limits_{t_i}^{t_{i+1}} F(s,\bar X_i,\bar y_i)ds\right)+\right.\right.$$

$$+h\left(\int\limits_{t_i}^{t_{i+1}} F(s,\bar X_i,\bar y_i)ds, \int\limits_{t_i}^{t_{i+1}} \bar F(s,\bar X_i,\bar y_i)ds\right)+h\left(\int\limits_{t_i}^{t_{i+1}} \bar F(s,\bar X_i,\bar y_i)ds, \int\limits_{t_i}^{t_{i+1}} \bar F(s,\bar X(s),\bar y(s))ds\right)\Bigg)+$$

$$+h\left(\int\limits_{t_k}^{t} F(s,\bar X(s),\bar y(s))ds, \int\limits_{t_k}^{t} F(s,\bar X_k,\bar y_k)ds\right)+h\left(\int\limits_{t_k}^{t} F(s,\bar X_k,\bar y_k)ds, \int\limits_{t_k}^{t} \bar F(s,\bar X_k,\bar y_k)ds\right)+$$

$$\left.+\,h\left(\int\limits_{t_k}^{t} \bar F(s,\bar X_k,\bar y_k)ds, \int\limits_{t_k}^{t} \bar F(s,\bar X(s),\bar y(s))ds\right)\right] \le$$

$$\le \varepsilon\left[\sum_{i=0}^{k-1}\left(\int\limits_{t_i}^{t_{i+1}} h\left(F(s,\bar X(s),\bar y(s)), F(s,\bar X_i,\bar y_i)\right)\,ds+\right.\right.$$

$$\left.+h\left(\int\limits_{t_i}^{t_{i+1}} F(s,\bar X_i,\bar y_i)ds, \int\limits_{t_i}^{t_{i+1}} \bar F(s,\bar X_i,\bar y_i)ds\right)+\int\limits_{t_i}^{t_{i+1}} h\left(\bar F(s,\bar X_i,\bar y_i), \bar F(s,\bar X(s),\bar y(s))\right)ds\right)+$$

$$+\int\limits_{t_k}^{t} h\left(F(s,\bar X(s),\bar y(s)), F(s,\bar X_k,\bar y_k)\right)ds+$$

$$\left.+h\left(\int\limits_{t_k}^{t} F(s,\bar X_k,\bar y_k)ds, \int\limits_{t_k}^{t} \bar F(s,\bar X_k,\bar y_k)ds\right)+\int\limits_{t_k}^{t} h\left(\bar F(s,\bar X_k,\bar y_k), \bar F(s,\bar X(s),\bar y(s))\right)ds\right] \le$$

$$\le \varepsilon\left[\sum_{i=0}^{k}\int\limits_{t_i}^{t_{i+1}}\left(h\left(F(s,\bar X(s),\bar y(s)), F(s,\bar X_i,\bar y_i)\right)+h\left(\bar F(s,\bar X(s),\bar y(s)), \bar F(s,\bar X_i,\bar y_i)\right)\right)ds+\right.$$

$$+\sum_{i=0}^{k-1} h\left(\int_{t_i}^{t_{i+1}} F(s,\bar{X}_i,\bar{y}_i)ds, \int_{t_i}^{t_{i+1}} \bar{F}(s,\bar{X}_i,\bar{y}_i)ds\right) + h\left(\int_{t_k}^{t} F(s,\bar{X}_k,\bar{y}_k)ds, \int_{t_k}^{t} \bar{F}(s,\bar{X}_k,\bar{y}_k)ds\right)\Bigg].$$

Similarly

$$\varepsilon\left\|\int_0^t \left[g(s,\bar{X}(s),\bar{y}(s)) - \bar{g}(s,\bar{X}(s),\bar{y}(s))\right] ds\right\| \le$$

$$\le \varepsilon\left[\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}}\left(\left\|g(s,\bar{X}(s),\bar{y}(s)) - g(s,\bar{X}_i,\bar{y}_i)\right\| + \left\|\bar{g}(s,\bar{X}(s),\bar{y}(s)) - \bar{g}(s,\bar{X}_i,\bar{y}_i)\right\|\right) ds + \right.$$

$$\left. +\sum_{i=0}^{k-1}\left\|\int_{t_i}^{t_{i+1}}\left(g(s,\bar{X}_i,\bar{y}_i) - \bar{g}(s,\bar{X}_i,\bar{y}_i)\right) ds\right\| + \left\|\int_{t_k}^{t}\left(g(s,\bar{X}_k,\bar{y}_k) - \bar{g}(s,\bar{X}_k,\bar{y}_k)\right) ds\right\|\right].$$

Notice that

$$h\left(\bar{X}(s),\bar{X}_i\right) = h\left(\bar{X}(s),\bar{X}(t_i)\right) \le \varepsilon\int_{t_i}^{s} h\left(\bar{F}(v,\bar{X}(v),\bar{y}(v)),\{0\}\right) dv \le \varepsilon M(s-t_i),$$

$$\|\bar{y}(s)-\bar{y}_i\| = \|\bar{y}(s)-\bar{y}(t_i)\| \le \varepsilon\int_{t_i}^{s}\left\|\bar{g}(v,\bar{X}(v),\bar{y}(v))\right\| dv \le \varepsilon M(s-t_i).$$

Then

$$\varepsilon\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}} h\left(F(s,\bar{X}(s),\bar{y}(s)),F(s,\bar{X}_i,\bar{y}_i)\right) ds \le$$

$$\le \varepsilon\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}} \lambda\left[h\left(\bar{X}(s),\bar{X}_i\right) + \|\bar{y}(s)-\bar{y}_i\|\right] ds \le$$

$$\le \varepsilon\lambda\cdot 2\varepsilon M\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}}(s-t_i)ds =$$

$$= 2\varepsilon^2\lambda M\sum_{i=0}^{k}\frac{(t_{i+1}-t_i)^2}{2} = \varepsilon^2\lambda M\cdot(k+1)\cdot\left(\frac{L}{\varepsilon m}\right)^2 \le \frac{\lambda ML^2}{m},$$

$$\varepsilon\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}} h\left(\bar F(s,\bar X(s),\bar y(s)),\bar F(s,\bar X_i,\bar y_i)\right) \le \frac{\lambda ML^2}{m},$$

$$\varepsilon\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}} \left\|g(s,\bar X(s),\bar y(s)) - g(s,\bar X_i,\bar y_i)\right\| ds \le$$

$$\le \varepsilon\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}} \lambda\left[h\left(\bar X(s),\bar X_i\right) + \|\bar y(s)-\bar y_i\|\right] ds \le$$

$$\le \varepsilon^2\lambda M\cdot(k+1)\cdot\left(\frac{L}{\varepsilon m}\right)^2 \le \frac{\lambda ML^2}{m},$$

$$\varepsilon\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}} \left\|\bar g(s,\bar X(s),\bar y(s)) - \bar g(s,\bar X_i,\bar y_i)\right\| \le \frac{\lambda ML^2}{m}.$$

Using condition 2) of the theorem there exist such monotone decreasing functions $f_1(t)$ and $f_2(t)$ that tend to zero as $t\to\infty$, that for all $(X,y)\in D_1\times D_2$ we have:

$$h\left(\int_0^t F(s,\bar X,\bar y)ds, \int_0^t \bar F(s,\bar X,\bar y)ds\right) \le t\cdot f_1(t),$$

$$\left\|\int_0^t \left(g(s,\bar X,\bar y) - \bar g(s,\bar X,\bar y)\right) ds\right\| \le t\cdot f_2(t).$$

Then

$$\varepsilon h\left(\int_{t_i}^{t_{i+1}} F(s,\bar X_i,\bar y_i)ds, \int_{t_i}^{t_{i+1}} \bar F(s,\bar X_i,\bar y_i)ds\right) =$$

$$= \varepsilon h\left(\int_0^{t_{i+1}} F(s,\bar X_i,\bar y_i)ds \overset{h}{-} \int_0^{t_i} F(s,\bar X_i,\bar y_i)ds, \int_0^{t_{i+1}} \bar F(s,\bar X_i,\bar y_i)ds \overset{h}{-} \int_0^{t_i} \bar F(s,\bar X_i,\bar y_i)ds\right) \le$$

$$\le \varepsilon\left[h\left(\int_0^{t_{i+1}} F(s,\bar X_i,\bar y_i)ds, \int_0^{t_{i+1}} \bar F(s,\bar X_i,\bar y_i)ds\right) + h\left(\int_0^{t_i} F(s,\bar X_i,\bar y_i)ds, \int_0^{t_i} \bar F(s,\bar X_i,\bar y_i)ds\right)\right] \le$$

$$\leq \varepsilon \left[t_{i+1} \cdot f_1(t_{i+1}) + t_i \cdot f_1(t_i)\right] \leq 2 \sup_{\tau \in [0,L]} \tau f_1\left(\frac{\tau}{\varepsilon}\right) = \gamma_1(\varepsilon),$$

$$\varepsilon \left\| \int_{t_i}^{t_{i+1}} \left(g(s, \bar{X}_i, \bar{y}_i) - \bar{g}(s, \bar{X}_i, \bar{y}_i)\right) ds \right\| =$$

$$= \varepsilon \left\| \int_0^{t_{i+1}} \left(g(s, \bar{X}_i, \bar{y}_i) - \bar{g}(s, \bar{X}_i, \bar{y}_i)\right) ds - \int_0^{t_i} \left(g(s, \bar{X}_i, \bar{y}_i) - \bar{g}(s, \bar{X}_i, \bar{y}_i)\right) ds \right\| \leq$$

$$\leq \varepsilon \left[\left\| \int_0^{t_{i+1}} \left(g(s, \bar{X}_i, \bar{y}_i) - \bar{g}(s, \bar{X}_i, \bar{y}_i)\right) ds \right\| + \left\| \int_0^{t_i} \left(g(s, \bar{X}_i, \bar{y}_i) - \bar{g}(s, \bar{X}_i, \bar{y}_i)\right) ds \right\| \right] \leq$$

$$\leq \varepsilon \left[t_{i+1} \cdot f_2(t_{i+1}) + t_i \cdot f_2(t_i)\right] \leq 2 \sup_{\tau \in [0,L]} \tau f_2\left(\frac{\tau}{\varepsilon}\right) = \gamma_2(\varepsilon),$$

where $\tau = \varepsilon t$, a $\lim\limits_{\varepsilon \to 0} \gamma_1(\varepsilon) = 0$, $\lim\limits_{\varepsilon \to 0} \gamma_2(\varepsilon) = 0$. Similarly

$$h\left(\int_{t_k}^{t} F(s, \bar{X}_k, \bar{y}_k) ds, \int_{t_k}^{t} \bar{F}(s, \bar{X}_k, \bar{y}_k) ds \right) \leq$$

$$\leq \varepsilon \left[t \cdot f_1(t) + t_k \cdot f_1(t_k)\right] \leq 2 \sup_{\tau \in [0,L]} \tau f_1\left(\frac{\tau}{\varepsilon}\right) = \gamma_1(\varepsilon),$$

$$\varepsilon \left\| \int_{t_k}^{t} \left(g(s, \bar{X}_k, \bar{y}_k) - \bar{g}(s, \bar{X}_k, \bar{y}_k)\right) ds \right\| \leq$$

$$\leq \varepsilon \left[t \cdot f_2(t) + t_k \cdot f_2(t_k)\right] \leq 2 \sup_{\tau \in [0,L]} \tau f_2\left(\frac{\tau}{\varepsilon}\right) = \gamma_2(\varepsilon).$$

So

$$\varepsilon h\left(\int_0^t F(s, \bar{X}(s), \bar{y}(s)) ds, \int_0^t \bar{F}(s, \bar{X}(s), \bar{y}(s)) ds \right) \leq \frac{2\lambda M L^2}{m} + (k+1)\gamma_1(\varepsilon) \leq$$

$$\leq \frac{2\lambda M L^2}{m} + m\gamma_1(\varepsilon) \equiv \phi_1(\varepsilon, m), \tag{10}$$

$$\varepsilon \left\| \int_0^t \left[g(s, \bar{X}(s), \bar{y}(s)) - \bar{g}(s, \bar{X}(s), \bar{y}(s))\right] ds \right\| \leq \frac{2\lambda M L^2}{m} + m\gamma_2(\varepsilon) \equiv \phi_2(\varepsilon, m). \tag{11}$$

If we substitute (10) in (8) and (11) in (9), we will get

$$h\left(X(t),\bar{X}(t)\right)\le\varepsilon\lambda\int\limits_0^t\left[h(X(s),\bar{X}(s))+\|y(s)-\bar{y}(s)\|\right]ds+\varphi_1(\varepsilon,m),$$

$$\|y(t)-\bar{y}(t)\|\le\varepsilon\lambda\int\limits_0^t\left[h(X(s),\bar{X}(s))+\|y(s)-\bar{y}(s)\|\right]ds+\varphi_2(\varepsilon,m).$$

Adding these two inequalities and applying Gronwall-Bellmann lemma we get

$$h\left(X(t),\bar{X}(t)\right)+\|y(t)-\bar{y}(t)\|\le e^{2\varepsilon\lambda\int\limits_0^t 1ds}\left(\phi_1(\varepsilon,m)+\phi_2(\varepsilon,m)\right)=$$
$$=e^{2\varepsilon\lambda t}\left(\frac{4\lambda ML^2}{m}+m\gamma_1(\varepsilon)+m\gamma_2(\varepsilon)\right)\le$$
$$\le e^{2\lambda L}\left(\frac{4\lambda ML^2}{m}+m\gamma_1(\varepsilon)+m\gamma_2(\varepsilon)\right).$$

Then for every summand the inequality holds:

$$h\left(X(t),\bar{X}(t)\right)\le e^{2\lambda L}\left(\frac{4\lambda ML^2}{m}+m\gamma_1(\varepsilon)+m\gamma_2(\varepsilon)\right),$$

$$\|y(t)-\bar{y}(t)\|\le e^{2\lambda L}\left(\frac{4\lambda ML^2}{m}+m\gamma_1(\varepsilon)+m\gamma_2(\varepsilon)\right).$$

Let $\eta_1=\min\{\eta,\rho,\xi\}$. Choose m to satisfy the inequality

$$e^{2\lambda L}\frac{\lambda ML^2}{m}<\frac{\eta_1}{12}.$$

Then fix m and choose $\varepsilon_0\in(0,\sigma]$ such that for $\varepsilon\in(0,\varepsilon_0]$ the inequalities hold

$$e^{2\lambda L}m\gamma_1(\varepsilon)\le\frac{\eta_1}{3},e^{2\lambda L}m\gamma_2(\varepsilon)\le\frac{\eta_1}{3}.$$

Then $h\left(X(t),\bar{X}(t)\right)\le\eta_1$ and $\|y(t)-\bar{y}(t)\|\le\eta_1$ if the solution $(X(t),y(t))$ belongs to the domain $D_1\times D_2$. And it follows from condition 3) of the theorem as $\eta_1=\min\{\eta,\rho,\xi\}$.

So, we get that for any $\eta>0$ and $L>0$ there exists such ε_0, that for $\varepsilon\in(0,\varepsilon_0]$ and $t\in[0,L\varepsilon^{-1}]$ the following inequalities fulfill

$$h\left(X(t),\bar{X}(t)\right)\le\eta,\|y(t)-\bar{y}(t)\|\le\eta.$$

The theorem is proved. □

3. Conclusion

This paper contains the substantiation of the scheme of partial averaging for one class of hybrid systems where one equation is a differential equation with Hukuhara derivative and the other one is an ordinary differential equation. In case when the right-hand sides are periodic in time one can obtain a better estimate. Namely one can show that for any $L > 0$ there exist $C(L) > 0$ and $\varepsilon_0(L) > 0$ such that the conclusion of the theorem holds with $\eta = C\varepsilon$.

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