GUARANTEED ESTIMATES OF LINEAR FUNCTIONALS ON VELOCITY OF A VISCOUS INCOMPRESSIBLE FLUID UNDER UNCERTAINTIES

© Olexander Nakonechny, Yuri Podlipenko

Taras Shevchenko National University of Kyiv
The Faculty of Cybernetics
e-mail: *yourip@mail.ru*

Abstract. *The creation and justification of the methods for guaranteed estimation of linear functionals from solutions to the boundary value problems for linearized stationary Navier-Stokes equations in bounded open Lipschitzian domains are considered.*

Introduction

Problems of optimal reconstruction of solutions of linearized stationary Navier-Stokes equations under incomplete data are investigated. These problems play an important role in mathematical physics. Depending on a character of an apriori information, stochastic or deterministic approach are possible. The choice is determined by nature of the parameters in the problem, which can be random or not. Moreover the optimality of estimations depends on a criterion with respect to which a given value is evaluated.

We assume that right-hand sides of linearized Navier-Stokes equations are unknown and belong to the given bounded subsets of the space of all square integrable functions in the considered domain and for solving the estimation problems we must have supplementary data (observations) depending on solutions of these equations. We suppose that observation errors (noises) are realizations of the stochastic fields, with unknown moment functions of the second order also belonging to certain given subsets.

Our approach is as follows. We are looking for linear with respect to observations optimal estimates of solutions of linearized Navier-Stokes equations from the condition of minimum of maximal mean square error of estimation taken over the above subsets.

We consider constructive methods for obtaining such estimates, which is expressed in terms of solutions of special variational equations.

Guaranteed estimation problems for some other types of ordinary and partial differential equations are investigated in [1]–[5].

1. Preliminaries and auxiliary results

If X is a Hilbert space over $\mathbb{R}$ with inner product $(\cdot,\cdot)_X$ and norm $\|\cdot\|_X$, then by $J_X \in \mathscr{L}(X,X')$ we will denote an operator, called a canonical isomorphism from X onto dual space X', and defined by the equality $(v,u)_X = < v, J_X u >_{X\times X'}$ $\forall u,v \in X$, where $< x, f >_{X\times X'} := f(x)$ for $x \in X$, $f \in X'$, and $\mathscr{L}(X, Y)$ is the set of bounded linear operators mapping X into a Hilbert space Y.

Further we use the following notations: $x = (x_1, \ldots, x_n)$ denotes a spatial variable that is varied in a bounded open Lipschitzian domain $D \subset \mathbb{R}^n$, with boundary Γ;

$dx = dx_1 \cdots dx_n$ is a Lebesgue measure in $\mathbb{R}^n$;

$\mathscr{D}(D)$ is the space of infinitely differentiable functions with compact support contained in D.

A continuous linear form on $\mathscr{D}(D)$ is called a distribution on D. We denote by $\mathscr{D}'(D)$ the set of distributions on D. If $T \in \mathscr{D}'(D)$ we denote by $< T, \phi >$ its value on the function $\phi \in \mathscr{D}(D)$.

If $T \in \mathscr{D}'(D)$ the derivative $D_i T = \frac{\partial T}{\partial x_i}$ which coincides with the usual differentiation of continuously differentiable functions, is defined by $< \frac{\partial T}{\partial x_i}, \phi >= - < T, \frac{\partial \phi}{\partial x_i} >$.

We denote by $L^2(D)$ the space of the real functions defined on D with the second power absolutely integrable for the Lebesgue measure dx. This is a Hilbert space with the norm

$$\|u\|_{L^2(D)} = \left(\int_D |u(x)|^2\, dx\right)^{1/2}.$$

and inner product

$$(u, v)_{L^2(D)} = \int_D u(x)v(x)\, dx.$$

The Sobolev space $H^1(D)$ is the space of functions in $L^2(D)$ with derivatives of order 1 also belonging to $L^2(D)$. This is a Hilbert space with the norm

$$\|u\|_{H^1(D)} = \left(\|u\|^2_{L^2(D)} + \sum_{j=1}^{n} \|D_j u\|^2_{L^2(D)}\right)^{1/2}$$

and inner product

$$(u, v)_{H^1(D)} = (u, v)_{L^2(D)} + \sum_{j=1}^{n} (D_j u, D_j v)_{L^2(D)}.$$

The closure of $\mathscr{D}(D)$ in $H^1(D)$ is denoted by $H^1_0(D)$.

We will also use the notation $\mathbf{L}^2(D) = \{L^2(D)\}^n$, $\mathbf{H}^1(D) = \{H^1(D)\}^n$, $\mathbf{H}^1_0(D) = \{H^1_0(D)\}^n$, $\boldsymbol{\mathscr{D}}(D) = \{\mathscr{D}(D)\}^n$, $\boldsymbol{\mathscr{D}}'(D) = \{\mathscr{D}'(D)\}^n$ for the product spaces consisting of vector functions $\mathbf{u} = (u_1, \ldots, u_n)$ whose componets belong to one of the spaces $L^2(D)$, $H^1(D)$, $H^1_0(D)$, $\mathscr{D}(D)$, $\mathscr{D}'(D)$ respectively, and we suppose that these product spaces are equipped with the usual product norm and inner product (except $\mathscr{D}(D)^n$ and $\mathscr{D}'(D)^n$ which are not normed spaces). For example, if

$\mathbf{u} = (u_1, \dots, u_n), \mathbf{v} = (v_1, \dots, v_n) \in \mathbf{L}^2(D)$ then

$$(\mathbf{u}, \mathbf{v})_{\mathbf{L}^2(D)} = \sum_{i=1}^{n} (u_i, v_i)_{L^2(D)}, \quad \|\mathbf{u}\|_{\mathbf{L}^2(D)} = (\mathbf{u}, \mathbf{u})_{\mathbf{L}^2(D)}^{1/2} = \Big\{ \sum_{i=1}^{n} \|u_i\|_{L^2(D)}^2 \Big\}^{1/2}.$$

For every $v \in \mathscr{D}'(D)$ we put

$$\mathbf{grad}\, v := \left(\frac{\partial v}{\partial x_1}, \dots, \frac{\partial v}{\partial x_n} \right),$$

which defines the linear differential operator denoted by **grad** from $\mathscr{D}'(D)$ to $\boldsymbol{\mathscr{D}}'(D)$.

We define the linear differential operator denoted by div from $\boldsymbol{\mathscr{D}}'(D)$ to $\mathscr{D}'(D)$ by

$$\operatorname{div} \mathbf{v} := \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i} \quad \forall \mathbf{v} = (v_1, \dots, v_n) \in \boldsymbol{\mathscr{D}}'(D)$$

and the Laplace operator Δ from $\boldsymbol{\mathscr{D}}'(D) \to \boldsymbol{\mathscr{D}}'(D)$ by

$$\Delta \mathbf{v} = \left(\sum_{i=1}^{n} \frac{\partial^2 v_1}{\partial x_i^2}, \dots, \sum_{i=1}^{n} \frac{\partial^2 v_n}{\partial x_i^2} \right).$$

Let $\mathscr{V} = \{\mathbf{u} \in \boldsymbol{\mathscr{D}}(D),\ \operatorname{div} \mathbf{u} = 0\}$ and V be the closure of $\mathscr{V}$ in $\mathbf{H}_0^1(D)$. In [8] it is shown that

$$V = \{\mathbf{u} \in \mathbf{H}_0^1(D),\ \operatorname{div} \mathbf{u} = 0\}.$$

The space V is a Hilbert space with the inner product

$$(\mathbf{u}, \mathbf{v})_V := \sum_{i=1}^{n} (D_i \mathbf{u}, D_i \mathbf{v})_{\mathbf{L}^2(D)} = \sum_{i=1}^{n} (\mathbf{grad}\, u_i, \mathbf{grad}\, v_i)_{\mathbf{L}^2(D)}$$

and norm $\|\mathbf{u}\|_V = (\mathbf{u}, \mathbf{u})_V^{1/2}$, where $D_i \mathbf{u} = (D_i u_1, \dots, D_i u_n)$.

We will also apply the generalized Schwarz's inequality (see, for example, [10], page 186):

$$(x, y)_X^2 \le (R^{-1} x, x)_X (R y, y)_X \quad \forall x, y \in X, \tag{1}$$

where $R : X \to X$ is a linear bounded self-adjoint positive definite operator in Hilbert space X over $\mathbb{R}$, and inequality (1) is transformed to an equality on the element $y = \lambda R^{-1} x,\ \ \forall \lambda \in \mathbb{R}$.

Let H be a separable Hilbert space over $\mathbb{R}$. By $L^2(\Omega, H)$ we denote the Bochner space composed of random* variables $\xi = \xi(\omega)$ defined on a certain probability space $(\Omega, \mathscr{B}, P)$

*Random variable ξ with values in Hilbert space H is considered as a function $\xi : \Omega \to H$ mapping random events $E \in \mathscr{B}$ to Borel sets in H (Borel σ-algebra in H is generated by open sets in H).

with values in H such that

$$\|\xi\|^2_{L^2(\Omega,H)} = \int_\Omega \|\xi(\omega)\|^2_H dP(\omega) < \infty. \tag{2}$$

In this case there exists the Bochner integral

$$\mathbb{E}\xi := \int_\Omega \xi(\omega)\, dP(\omega) \in H \tag{3}$$

called the expectation or the mean value of random variable $\xi(\omega)$ which satisfies the condition

$$(h, \mathbb{E}\xi)_H = \int_\Omega (h, \xi(\omega))_H \, dP(\omega) \quad \forall h \in H. \tag{4}$$

Being applied to random variable ξ with values in $\mathbb{R}$ this expression leads to a usual definition of its expectation because the Bochner integral (3) reduces to a Lebesgue integral with probability measure $dP(\omega)$.

In $L^2(\Omega, H)$ one can introduce the inner product

$$(\xi, \eta)_{L^2(\Omega,H)} := \int_\Omega (\xi(\omega), \eta(\omega))_H \, dP(\omega) \quad \forall \xi, \eta \in L^2(\Omega, H). \tag{5}$$

Applying the sign of expectation, one can write relationships (2), (4), (5) as

$$\|\xi\|^2_{L^2(\Omega,H)} = \mathbb{E}\|\xi(\omega)\|^2_H, \tag{6}$$

$$(h, \mathbb{E}\xi)_H = \mathbb{E}(h, \xi(\omega))_H \quad \forall h \in H, \tag{7}$$

$$(\xi, \eta)_{L^2(\Omega,H)} := \mathbb{E}(\xi(\omega), \eta(\omega))_H \quad \forall \xi, \eta \in L^2(\Omega, H). \tag{8}$$

$L^2(\Omega, H)$ equipped with norm (6) and inner product (8) is a Hilbert space.

The Stokes problem consists of finding a vector function $\mathbf{v} = (v_1, \ldots, v_n) : D \to \mathbb{R}^n$ and a scalar function $p : D \to \mathbb{R}$ from equations

In this paper we focus on the estimation problems for linearized Navier-Stokes equations

$$-\nu \boldsymbol{\Delta} \mathbf{v} + \operatorname{grad} p = \mathbf{f} \quad \text{в} \quad D, \tag{9}$$

$$\operatorname{div} \mathbf{v} = 0 \quad \text{в} \quad D, \tag{10}$$

$$\mathbf{v} = 0 \quad \text{на} \quad \Gamma, \tag{11}$$

that simulate the motion of a viscous incompressible fluid in the domain D. Here vector-functions $\mathbf{v} = (v_1, \ldots, v_n), \mathbf{f} = (f_1, \ldots, f_n) : D \to \mathbb{R}^n$, and scalar function $p : D \to \mathbb{R}$ represent the velocity, body force, and the pressure fields, respectively, and the positive constant ν is the coefficient of kinematic viscosity.

It is known that in the case, when $\mathbf{f} \in \mathbf{L}^2(D)$, vector function $\mathbf{v}$ can be found from the following equations

$$\mathbf{v} \in V, \tag{12}$$

$$\nu \sum_{i=1}^{n} (D_i \mathbf{v}, D_i \mathbf{u})_{\mathbf{L}^2(D)} = (\mathbf{f}, \mathbf{u})_{\mathbf{L}^2(D)} \quad \forall \mathbf{u} \in V. \tag{13}$$

Problem (12)–(13), called the variational statement of the Stokes problem (9)–(11), is uniquely solvable [6]–[9].

Since in this paper, from observations of velocity $\mathbf{v}$ only the linear functionals of the form $l(\mathbf{v})$ will be evaluated, in future we will deal only with the variational statement (12)– (13) of the Stokes problem (9)– (11).

2. Setting of the estimation problem

The estimation problem consists in the following: from the observations

$$y = C\mathbf{v} + \xi, \tag{14}$$

find optimal in a certain sense estimate of the functional

$$l(\mathbf{v}) = (\mathbf{l}_0, \mathbf{v})_{\mathbf{L}^2(D)} = \int_D (\mathbf{l}_0(x), \mathbf{v}(x))_{\mathbb{R}^n}\, dx \tag{15}$$

in the class of estimates linear w.r.t. observations (14),

$$\widehat{l(\mathbf{v})} = (u, y)_H + c, \tag{16}$$

under the assumption that errors $\xi = \xi(\omega)$ in observations (14) are realizations of random variables defined on a certain probability space $(\Omega, \mathscr{B}, P)$ with values in a separable Hilbert space H over $\mathbb{R}$, belong to the set G_1, and $\mathbf{f} \in G_0$, where

$$G_0 = \left\{ \tilde{\mathbf{f}} : \tilde{\mathbf{f}} \in \mathbf{L}^2(D), (Q\tilde{\mathbf{f}} - \mathbf{f}_0, \tilde{\mathbf{f}} - \mathbf{f}_0)_{\mathbf{L}^2(D)} \le \varepsilon_0 \right\}, \tag{17}$$

$$G_1 = \{ \tilde{\xi} : \tilde{\xi} \in L^2(\Omega, H),\ \mathbb{E}\tilde{\xi} = 0,\ \mathbb{E}(Q_1\tilde{\xi}, \tilde{\xi})_H \le \varepsilon_1 \}. \tag{18}$$

Here $\epsilon_k > 0$, $k = 0, 1$, are given constants; $u \in H$; $c \in \mathbb{R}$; $(\cdot, \cdot)_H$ is inner product in H; $\mathbf{l}_0, \mathbf{f}_0 \in \mathbf{L}^2(D)$ are given real-valued functions; $C \in \mathscr{L}(\mathbf{L}^2(D), H)$ is linear continuous operator; and Q, Q_1, are self-adjoint positive definite operators in $\mathbf{L}^2(D)$ and H, respectively, for which there exist bounded inverse operators Q^{-1} and Q_1^{-1}. Further, without loss of generality we may set $\epsilon_k = 1$, $k = 0, 1$.

Definition 1. An estimate

$$\widehat{\widehat{l(\mathbf{v})}} = (\hat{u}, y)_H + \hat{c}$$

is called a minimax (or a guaranteed) estimate of $l(\mathbf{v})$ if element $\hat{u} \in H$ and a number $\hat{c} \in \mathbb{R}$ are determined from the condition

$$\inf_{u \in H,\, c \in \mathbb{R}} \sigma(u, c) = \sigma(\hat{u}, \hat{c}),$$

where

$$\sigma(u, c) := \sup_{\tilde{\mathbf{f}} \in G_0,\, \tilde{\xi} \in G_1} \mathbb{E}[l(\tilde{\mathbf{v}}) - \widehat{l(\tilde{\mathbf{v}})}]^2,$$

$\tilde{\mathbf{v}}$ is a solution to problem (12) –(13) when $\mathbf{f} = \tilde{\mathbf{f}}$, $\widehat{l(\tilde{\mathbf{v}})} = (u, \tilde{y})_H + c$, $\tilde{y} = C\tilde{\mathbf{v}} + \tilde{\xi}$.

The quantity $\sigma := [\sigma(\hat{u}, \hat{c})]^{1/2}$ is called the error of the minimax estimation of $l(\mathbf{v})$.

Thus, the minimax estimate is an estimate minimizing the maximal mean-square estimation error calculated for the "worst" implementation of perturbations.

3. Reducing of the estimation problem to the optimal control problem

To find representations for minimax estimates, we first reduce this problem to certain optimal control problem.

For every fixed $u \in H$ introduce a function $\mathbf{z}(x; u)$, as a solution to the following variational problem:

$$\mathbf{z}(\cdot; u) \in V, \tag{19}$$

$$\nu \sum_{i=1}^{n} (D_i \mathbf{z}(\cdot; u), D_i \mathbf{u})_{\mathbf{L}^2(D)} = (\mathbf{l}_0 - C^* J_H u, \mathbf{u})_{\mathbf{L}^2(D)} \quad \forall \mathbf{u} \in V, \tag{20}$$

where $C^* : H' \to \mathbf{L}^2(D)$ is an operator adjoint of C defined by

$$(p, C^* g)_{\mathbf{L}^2(D)} = < Cp, g >_{H \times H'} \quad \forall p \in \mathbf{L}^2(D), g \in H'.$$

Then the following assertion is valid.

Lemma 1. *The problem of minimax estimation of* $l(\mathbf{v})$ *(i.e. the determination of* $\hat{u}$ *and* $\hat{c}$*) is equivalent to the problem of optimal control of the system described by equation (19),* (20) *with a cost function*

$$I(u) = (Q^{-1} \mathbf{z}(\cdot; u), \mathbf{z}(\cdot; u))_{\mathbf{L}^2(D)} + (Q_1^{-1} u, u)_H \to \inf_{u \in H}. \tag{21}$$

Proof. From relation (15) and (16) at $\mathbf{v} = \tilde{\mathbf{v}}$, we have

$$l(\tilde{\mathbf{v}}) - \widehat{l(\tilde{\mathbf{v}})} = \int_D (\mathbf{l}_0(x), \tilde{\mathbf{v}}(x))_{\mathbb{R}^3}\, dx - \int_D (C^* J_H u(x), \tilde{\mathbf{v}}(x))_{\mathbb{R}^n}\, dx - (u, \tilde{\xi})_H - c$$

$$= \nu \sum_{i=1}^{n} (D_i \mathbf{z}(\cdot; u), D_i \tilde{\mathbf{v}})_{\mathbf{L}^2(D)} - (u, \tilde{\xi})_H - c. \tag{22}$$

Taking into account the fact, that $\tilde{\mathbf{v}}$ is a solution of problem (12) –(13) at $\mathbf{f} = \tilde{\mathbf{f}}$, and setting in (13) $\mathbf{u} = \mathbf{z}(\cdot; u)$, we come to the equality

$$\nu \sum_{i=1}^{n} (D_i \tilde{\mathbf{v}}, D_i \mathbf{z}(\cdot; u))_{\mathbf{L}^2(D)} = \int_D (\tilde{\mathbf{f}}(x), \mathbf{z}(x; u))_{\mathbb{R}^n}\, dx.$$

From (22) and the latter formula, we obtain

$$l(\tilde{\mathbf{v}}) - \widehat{l(\tilde{\mathbf{v}})} = \int_D (\tilde{\mathbf{f}}(x), \mathbf{z}(x; u))_{\mathbb{R}^n}\, dx - (u, \tilde{\xi})_H - c. \tag{23}$$

Applying to the right hand side of (23) the generalized Schwarz's inequality and (6)–(8), (17), (18), we find

$$\inf_{c \in \mathbb{R}^1} \sup_{\tilde{\mathbf{f}} \in G_0, \tilde{\xi} \in G_1} \mathbb{E}[l(\tilde{\mathbf{v}} - \widehat{l(\tilde{\mathbf{v}})}]^2 = \inf_{c \in \mathbb{R}^1} \sup_{\tilde{\mathbf{f}} \in G_0} \left\{ \int_D (\tilde{\mathbf{f}}(x), \tilde{\mathbf{z}}(x; u))_{\mathbb{R}^n}\, dx - c \right\}^2$$

$$+ \sup_{\tilde{\xi} \in G_1} \mathbb{E}\{(u, \tilde{\xi})_H\}^2 = \int_D (Q^{-1} \mathbf{z}(\cdot; u)(x), \mathbf{z}(x; u))_{\mathbb{R}^n}\, dx + (Q_1^{-1} u, u)_H.$$

with $c = \int_D (\mathbf{z}(x; u), \mathbf{f}_0(x))_{\mathbb{R}^n} dx$. The lemma is proved. □

4. Representation of guaranteed estimates of functionals from solutions of Stokes problem

Solving the optimal control problem (19) – (21) and applying arguments completely analogous to that used in the proof of Theorem 2 on page 62 from [2] , we prove the following.

Theorem 1. *The minimax estimate of* $l(\mathbf{v})$ *has the form*

$$\widehat{\widehat{l(\mathbf{v})}} = (\hat{u}, y)_H + \hat{c} = l(\hat{\mathbf{v}}) = \int_D (\mathbf{l}_0(x), \hat{\mathbf{v}})_{\mathbb{R}^n}(x)\, dx, \tag{24}$$

where

$$\hat{c} = \int_D (\hat{\mathbf{z}}(x), \mathbf{f}_0(x))_{\mathbb{R}^n}\, dx, \quad \hat{u} = Q_1 C \mathbf{p}, \tag{25}$$

the functions $\mathbf{p}(x)$ *and* $\hat{\mathbf{z}}(x)$ *are determined as a solution of the following problem:*

$$\hat{\mathbf{z}} \in V, \tag{26}$$

$$\nu \sum_{i=1}^{n} (D_i \hat{\mathbf{z}}, D_i \mathbf{u})_{\mathbf{L}^2(D)} = (\mathbf{l}_0 - C^* J_H Q_1 C \mathbf{p}, \mathbf{u})_{\mathbf{L}^2(D)} \quad \forall \mathbf{u} \in V, \tag{27}$$

$$\mathbf{p} \in V, \tag{28}$$

$$\nu \sum_{i=1}^{n} (D_i \mathbf{p}, D_i \mathbf{u})_{\mathbf{L}^2(D)} = (Q^{-1}\hat{\mathbf{z}}, \mathbf{u})_{\mathbf{L}^2(D)} \quad \forall \mathbf{u} \in V, \tag{29}$$

and the function $\hat{\mathbf{v}}$ *is determined from solution of the problem*

$$\hat{\mathbf{p}} \in V, \tag{30}$$

$$\nu \sum_{i=1}^{n} (D_i \hat{\mathbf{p}}, D_i \mathbf{u})_{\mathbf{L}^2(D)} = (C^* J_H Q_1 (y - C\hat{\mathbf{v}}), \mathbf{u})_{\mathbf{L}^2(D)} \quad \forall \mathbf{u} \in V, \tag{31}$$

$$\hat{\mathbf{v}} \in V, \tag{32}$$

$$\nu \sum_{i=1}^{n} (D_i \hat{\mathbf{v}}, D_i \mathbf{u})_{\mathbf{L}^2(D)} = (Q^{-1}\hat{\mathbf{p}} + \mathbf{f}_0, \mathbf{u})_{\mathbf{L}^2(D)} \quad \forall \mathbf{u} \in V. \tag{33}$$

Problems (26)–(29) and (30)–(33) are uniquely solvable.

The error of estimation σ *is given by an expression*

$$\sigma = [l(\mathbf{p})]^{1/2} = \left[\int_D (\mathbf{l}_0(x), \mathbf{p}(x))_{\mathbb{R}^n} \, dx \right]^{1/2}. \tag{34}$$

Note that the function $\hat{\mathbf{z}}(x) = \mathbf{z}(x; \hat{u})$, where $\mathbf{z}(x; u)$ is a solution of problem (19), (20), and $u = \hat{u} \in H$ is optimal control of the system governed by these equations with cost function (21) (see Lemma 1).

Also, as one can see from equations (30)–(33), the function $\hat{\mathbf{v}}$ entering into the representation $\widehat{\widehat{l(\mathbf{v})}} = l(\hat{\mathbf{v}})$ does not depend on the concrete functional l and, hence, can be taken as a good estimate of an unknown solution $\mathbf{v}$ of the problem (12)–(13).

5. Approximate guaranteed estimates of linear functionals from solutions of Stokes problem. Theorems on convergence

Using the Galerkin method for solving problems (26)–(29) and (30)–(33), we obtain approximate guaranteed estimates via solutions of linear algebraic equations and show their convergence to the optimal estimates.

Introduce a sequence of finite-dimensional subspaces V^h in the space V, defined by an infinite set of parameters $h_1, h_2, \ldots$ with $\lim_{k \to 0} h_k = 0$.

We say that sequence $\{V^h\}$ is complete in V, if for any $\mathbf{v} \in V$ and $\epsilon > 0$ there exists an $\hat{h} = \hat{h}(\mathbf{v}, \epsilon) > 0$ such that $\inf_{\mathbf{w} \in V^h} \|\mathbf{v} - \mathbf{w}\|_V < \varepsilon$ for any $h < \hat{h}$. In other words, the completeness of sequence $\{V^h\}$ means that any element $\mathbf{v} \in V$ may be approximated with any degree of accuracy by elements of $\{V^h\}$.

Such finite-dimensional subspaces V^h are constructed, for example, in [8], Ch 1, §4.

Take an approximate minimax estimate of $l(\mathbf{v})$ as

$$\widehat{l^h(\mathbf{v})} = (\hat{u}^h, y)_H + \hat{c}^h,$$

where

$$\hat{c}^h = \int_D (\hat{\mathbf{z}}^h(x), \mathbf{f}_0(x))_{\mathbb{R}^n}\, dx, \quad \hat{u}^h = Q_1 C \mathbf{p}^h, \tag{35}$$

and functions $\mathbf{p}^h(x)$, $\hat{\mathbf{z}}^h(x)$ are determined from the following system of variational equations:

$$\hat{\mathbf{z}}^h \in V^h, \tag{36}$$

$$\nu \sum_{i=1}^{n} (D_i \hat{\mathbf{z}}^h, D_i \mathbf{u}^h)_{\mathbf{L}^2(D)} = (\mathbf{l}_0 - C^* J_H Q_1 C \mathbf{p}^h, \mathbf{u}^h)_{\mathbf{L}^2(D)} \quad \forall \mathbf{u}^h \in V^h, \tag{37}$$

$$\mathbf{p}^h \in V, \tag{38}$$

$$\nu \sum_{i=1}^{n} (D_i \mathbf{p}^h, D_i \mathbf{u}^h)_{\mathbf{L}^2(D)} = (Q^{-1} \hat{\mathbf{z}}^h, \mathbf{u}^h)_{\mathbf{L}^2(D)} \quad \forall \mathbf{u}^h \in V^h. \tag{39}$$

The unique solvability of this system (and system (40)–(43)) follows from the proof of Theorem 1 in which V is replaced by V^h.

Theorem 2. *Approximate minimax estimate* $\widehat{l^h(\mathbf{v})}$ *of* $l(\mathbf{v})$ *tends to a minimax estimate* $\widehat{\widehat{l(\mathbf{v})}}$ *of this expression as* $h \to 0$ *in the sense that*

$$\lim_{h\to 0} \sup_{\tilde{\mathbf{f}}\in G_0,\, \tilde{\xi}\in G_1} \mathbb{E}|\widehat{l^h(\tilde{\mathbf{v}})} - \widehat{\widehat{l(\tilde{\mathbf{v}})}}|^2 = 0$$

and

$$\lim_{h\to 0} \sup_{\tilde{\mathbf{f}}\in G_0,\, \tilde{\xi}\in G_1} \mathbb{E}|\widehat{l^h(\tilde{\mathbf{v}})} - l(\tilde{\mathbf{v}})|^2 = \sup_{\tilde{\mathbf{f}}\in G_0,\, \tilde{\xi}\in G_1} \mathbb{E}|\widehat{\widehat{l(\tilde{\mathbf{v}})}} - l(\tilde{\mathbf{v}}))|^2,$$

where $\tilde{\mathbf{v}}$ *is a solution of problem* (12) –(13) *at* $\mathbf{f} = \tilde{\mathbf{f}}$, $\widehat{l^h(\tilde{\mathbf{v}})} = (u^h, \tilde{y})_H + c^h$, $\tilde{y} = C\tilde{\mathbf{v}} + \tilde{\xi}$.

Now, we formulate an analogous result for the case when an estimate $\hat{\mathbf{v}}$ of $\mathbf{v}$ is directly determined from solution to the problem (30)–(33). Namely, the following result holds.

Theorem 3. *Let* $\hat{\mathbf{v}}^h \in V^h$ *be an approximate estimate of the vector-function* $\hat{\mathbf{v}} \in V$ *determined from the solution to the variational problem*

$$\hat{\mathbf{p}}^h \in V^h, \tag{40}$$

$$\nu \sum_{i=1}^{n} (D_i \hat{\mathbf{p}}^h, D_i \mathbf{u}^h)_{\mathbf{L}^2(D)} = (C^* J_H Q_1 (y - C\hat{\mathbf{v}}^h), \mathbf{u}^h)_{\mathbf{L}^2(D)} \quad \forall \mathbf{u}^h \in V^h, \tag{41}$$

$$\hat{\mathbf{v}}^h \in V^h, \tag{42}$$

$$\nu \sum_{i=1}^{n} (D_i \hat{\mathbf{v}}^h, D_i \mathbf{u}^h)_{\mathbf{L}^2(D)} = (Q^{-1}\hat{\mathbf{p}}^h + \mathbf{f}_0, \mathbf{u}^h)_{\mathbf{L}^2(D)} \quad \forall \mathbf{u}^h \in V^h. \tag{43}$$

Then

$$\|\hat{\mathbf{v}} - \hat{\mathbf{v}}^h\|_V \to 0 \quad \text{as} \quad h \to 0$$

and the approximate minimax estimate $\widehat{l^h(\mathbf{v})}$ *of* $l(\mathbf{v})$ *has the form*

$$\widehat{l^h(\mathbf{v})} = l(\hat{\mathbf{v}}^h) = \int_D (\mathbf{l}_0(x), \hat{\mathbf{v}}^h(x))_{\mathbb{R}^n}\, dx. \tag{44}$$

The proofs of Theorem 2 and Theorem 3 are similary to the proof of Proposition 3.2 on page 32 from [1].

Introducing the basis in the space V^h, problems (36)–(39) i (40)–(43) can be rewritten as a systems of liner algebraic equations. To do this, let us denote the elements of the basis by $\boldsymbol{\xi}_i$ $(i = 1, \ldots, N)$ where $N = \dim V^h$. The fact that $\hat{\mathbf{z}}^h, \mathbf{p}^h, \hat{\mathbf{p}}^h, \hat{\mathbf{v}}^h$ belong to the space V^h means the existence of constants $\hat{z}_j$, p_j and $\hat{p}_j$, $\hat{v}_j$ such that

$$\hat{\mathbf{z}}^h = \sum_{j=1}^{N} \hat{z}_j \boldsymbol{\xi}_j, \quad \mathbf{p}^h = \sum_{j=1}^{N} p_j \boldsymbol{\xi}_j \tag{45}$$

and

$$\hat{\mathbf{p}}^h = \sum_{j=1}^{N} \hat{p}_j \boldsymbol{\xi}_j, \quad \hat{\mathbf{v}}^h = \sum_{j=1}^{N} \hat{v}_j^{(2)} \boldsymbol{\xi}_j. \tag{46}$$

Setting in (37) and (39) and in (41) and (43) $\mathbf{u}^h = \boldsymbol{\xi}_i$ $(i = 1, \ldots, N)$, we obtain that finding $\hat{\mathbf{z}}^h$, $\mathbf{p}^h$ and $\hat{\mathbf{p}}^h$, $\hat{\mathbf{v}}^h$ is equivalent to solving the following systems of linear algebraic equations with respect to coefficients $\hat{z}_j$, p_j and $\hat{p}_j$, $\hat{v}_j$ of expansions (45) and (46):

$$\sum_{j=1}^{N} a_{jl} \hat{z}_j + \sum_{j=1}^{N} a_{jl}^{(1)} p_j = b_l, \quad l = 1, \ldots, N, \tag{47}$$

$$\sum_{j=1}^{N} a_{il} p_j + \sum_{j=1}^{N} a_{jl}^{(2)} \hat{z}_j = 0, \quad l = 1, \ldots, N \tag{48}$$

and

$$\sum_{j=1}^{N} a_{jl} \hat{p}_j + \sum_{j=1}^{N} a_{jl}^{(1)} \hat{v}_j = b_l^{(1)}, \quad l = 1, \ldots, N, \tag{49}$$

$$\sum_{j=1}^{N} a_{il} \hat{v}_j + \sum_{j=1}^{N} a_{jl}^{(2)} \hat{p}_j = b_l^{(2)}, \quad l = 1, \ldots, N, \tag{50}$$

where

$$a_{jl} = \nu \sum_{i=1}^{n} (D_i \boldsymbol{\xi}_j, D_i \boldsymbol{\xi}_l)_{\mathbf{L}^2(D)}, \quad j, l = 1, \ldots, N, \tag{51}$$

$$a_{jl}^{(1)} = (C^* J_H Q_1 C \boldsymbol{\xi}_j, \boldsymbol{\xi}_l)_{\mathbf{L}^2(D)}, \quad j, l = 1, \ldots, N, \tag{52}$$

$$a_{jl}^{(2)} = -(Q^{-1} \boldsymbol{\xi}_j, \boldsymbol{\xi}_l)_{\mathbf{L}^2(D)}, \quad j, l = 1, \ldots, N, \tag{53}$$

$$b_l = (\mathbf{l}_0, \boldsymbol{\xi}_l)_{\mathbf{L}^2(D)}, \quad l = 1, \ldots, N, \tag{54}$$

$$b_l^{(1)} = (C^* J_H Q_1 y, \boldsymbol{\xi}_l)_{\mathbf{L}^2(D)}, \quad l = 1, \ldots, N, \tag{55}$$

$$b_l^{(2)} = (Q^{-1} \mathbf{f}_0, \boldsymbol{\xi}_l)_{\mathbf{L}^2(D)}, \quad l = 1, \ldots, N. \tag{56}$$

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