ON REALIZING PRESCRIBED QUALITY OF A CONTROLLED SYSTEM'S PROCESS UNDER UNCERTAINTLY

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Abstract. *In this paper, we discuss a method of auxiliary controlled models and the application of this method to solving some problems of robust control for differential equations. As objects for the approbation of the method, a system of nonlinear differential equations describing some ecological and economic processes is used. A solving algorithm, which is stable with respect to informational noises and computational errors, is presented.*

1. Introduction. Statement of the problem

A dynamical model connecting main economic and climatic indices was suggested in [5]. This model is oriented to developing an economic strategy directed to deceleration of global warming. The main goal of the model analysis is to provide the means for tackling the following question: whether the reduction of emissions of greenhouse gases is justified from the economical viewpoint or not. The model takes into account global processes: it is assumed that the structure of economy is the same for all countries; the climate change is characterized by the average value of the temperature on Earth's surface and so on. This model contains three types of parameters.

1) Constant parameters (their list is presented in tables 2.3 and 2.4 on page 21, [5]).
2) Functions that are considered (for simplicity of the analysis) as exogenous with respect to the model and are a priori given.
3) Inner functions that are connected to one another and to exogenous parameters by means of some algebraic and differential equations. The list of these functions is presented in table 2.3. (see [5]), and the model equations are presented in table 2.2. Let us give the list of functions:
   - : $\mu(t)$ is a rate of emissions reduction with respect to uncontrollable emissions,
   - : $E(t)$ is an amount of emissions of greenhouse gases, below GHGs (CO_2 (carbonic acid gas) and chlorine-fluorine carbons only),
   - : $M_1(t) = (M(t) - 590)$ is an excess of the mass of GHGs in the atmosphere in comparison with the pre-industrial period,
   - : $T_0(t)$ is an average atmospheric temperature (on Earth's surface),

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: $T_1(t)$ is an average deep-ocean temperature,
: $I(t)$ is a gross investment,
: $K(t)$ is a capital stock,
: $F(t)$ is an atmospheric radiative forcing from GHGs,
: $O(t)$ is a forcing of exogenous GHGs (i.e., of gases, which are considered as uncontrollable; there are all GHGs, besides CO_2 (carbonic acid gas) and chlorine-fluorine carbons),
: $A(t)$ is a level of technology,
: $\sigma(t)$ is the ratio of GHGs emissions to global output,
: $L(t)$ is a population at time t, also equal to labor inputs,
: $Q(t)$ is a gross world product.

Schematically, the connections between the inner functions can be pictured in the following way:

$$
\begin{array}{ccccccc}
T_0^{**} & \longrightarrow & \Omega & & & & \\
\uparrow & & \uparrow & \searrow & & & \\
F & & \mu & & Q & & \\
\uparrow & & \downarrow & \swarrow & \uparrow & \nwarrow & \\
M_1^{*} & \longleftarrow & E & & K^{*} & \longleftarrow & I
\end{array}
$$

Here the functions marked by asterisk are solutions of linear differential equations of the first order, the function $T_0(t)$ is a solution of a linear differential equation of the second order.

If we pass from the discrete model suggested by the authors to the "continuous" one, then the equations of the model Σ take the form:

$$
\begin{array}{rcl}
\dot{T}_0(t) & = & c_1T_0(t) + c_2T_1(t) + c_3F(t), \quad t \in [0, \vartheta] \\
\dot{T}_1(t) & = & c_4(T_0(t) - T_1(t)) \\
\dot{M}_1(t) & = & \beta E(t) - \delta_M M_1(t) \\
\dot{K}(t) & = & -\delta_K K(t) + I(t),
\end{array}
\tag{1}
$$

where t is time, ϑ is a terminal time moment,

$$F(t) = 4,1 \cdot \log_2 \left(1 + \frac{M_1(t)}{590}\right) + O(t),$$

$$E(t) = (1 - \mu(t))\sigma(t)Q(t),$$

$$Q(t) = (1 - b_1\mu(t)^{b_2})/(1 + \theta_1 T_0(t)^{\theta_2})A(t)K(t)^{\gamma}L(t)^{1-\gamma}.$$

An initial state of Σ, $x(0) = \{T_0(0), T_1(0), M_1(0), K(0))\}$, is assumed to be known and a priori given. It is natural to set $T_0(0) > 0$, $T_1(0) > 0$, and $K(0) > 0$. Functions $\mu(\cdot)$

and $I(\cdot)$ are considered as control parameters determining a strategy of global control of climate and economy. The numerical analysis of the model is performed in [5]. At that, the direct problem is solved, namely, possible strategies (rules of forming $\mu(\cdot)$ and $I(\cdot)$) are specified, and system's dynamics is computed. The comparative analysis of simulation results for different structures is performed. In addition, the analysis of sensitivity of results with respect to some model parameters is fulfilled.

In what follows, values μ and I, according to [5], are treated as controls and are denoted by the symbol u, i.e., $u = \{\mu, I\}$. We transform system (1) to the form (neglecting small values ($b_1 = 0,0686$, $\vartheta_1 = 0,00144$))

$$\begin{aligned} \dot{T}_0(t) &= c_1T_0(t) + c_2T_1(t) + 4,1c_3 \cdot \log_2\left(1 + \frac{M_1(t)}{590}\right) + c_3O(t), \quad t \in [0,\vartheta] \\ \dot{T}_1(t) &= c_4(T_0(t) - T_1(t)) \\ \dot{M}_1(t) &= E_1(t)(1 - \mu(t)) - \delta_M M_1(t) \\ \dot{K}(t) &= -\delta_K K(t) + I(t), \end{aligned} \tag{2}$$

where

$$E_1(t) = E_1(t, K) = \beta\sigma(t)A(t)K(t)^{\gamma}L(t)^{1-\gamma}.$$

Hereinafter, we consider the system Σ of form (2). The symbol $x(\cdot) = x(\cdot; x(0), u(\cdot))$ stands for the solution of system (2) with an initial state $x(0)$ and a control $u(\cdot) = \{\mu(\cdot), I(\cdot)\}$.

Our aim differs from the aim of [5]. We consider an "inverse" problem; its essence consists in the following. Some system's dynamics, i.e., a function $x_*(\cdot) = \{T_{0*}(\cdot), T_{1*}(\cdot), K_*(\cdot), M_{1*}(\cdot)\}$ generated by some unknown controls $\mu = \mu_*(\cdot)$ and $I = I_*(\cdot)$ is given. These controls may be program or feedback controls; the latter is formed, for example, by the rule $\mu_*(t) = \mu(t, x_*(t))$, $I_*(t) = I(t, x_*(t))$. Thus, the functions $x_*(\cdot) = \{T_{0*}(\cdot), T_{1*}(\cdot), K_*(\cdot), M_{1*}(\cdot)\}$ satisfy the system of equations

$$\begin{aligned} \dot{T}_{0*}(t) &= c_1T_{0*}(t) + c_2T_{1*}(t) + 4,1c_3 \cdot \log_2\left(1 + \frac{M_{1*}(t)}{590}\right) + c_3O(t), \quad t \in [0,\vartheta] \\ \dot{T}_{1*}(t) &= c_4(T_{0*}(t) - T_{1*}(t)) \\ \dot{M}_{1*}(t) &= E_{1*}(t, K_*)(1 - \mu_*(t)) - \delta_M M_{1*}(t) \\ \dot{K}_*(t) &= -\delta_K K_*(t) + I_*(t), \end{aligned} \tag{3}$$

where, emphasize once again, the functions $\mu_*(\cdot)$ and $I_*(\cdot)$ are unknown. It is known only that they are subject to restrictions of the form

$$I_*(t) \in [I_-, I_+], \quad \mu_*(t) \in [f_-, f_+] \quad \text{при } t \in [0, \vartheta]. \tag{4}$$

Here

$$-\infty < f_- < f_+ < +\infty, \quad 0 \le I_- < I_+ < +\infty.$$

The initial state of system (3), $x_*(0) = \{T_{0*}(0), T_{1*}(0), M_{1*}(0), K_*(0)\}$, is assumed to be $x(0)$.

The problem under consideration may be formulated in the following way. At frequent enough time moments

$$\tau_i \in \Delta = \{\tau_i\}_{i=0}^m, \quad \tau_{i+1} = \tau_i + \delta, \quad \tau_0 = 0, \quad \tau_m = \vartheta,$$

values of $T_0(\tau_i)$, $T_1(\tau_i)$, and $K(\tau_i)$ are inaccurately measured. Results of measurements (vectors $\{\xi_{1i}^h, \xi_{2i}^h, \xi_{3i}^h\} \in R^3$) satisfy the inequalities

$$|T_0(\tau_i) - \xi_{1i}^h|^2 + |T_1(\tau_i) - \xi_{2i}^h|^2 + |K(\tau_i) - \xi_{3i}^h|^2 \le h^2, \tag{5}$$

where $h \in (0,1)$ is a level of informational noise. Here and below, the symbol $|\cdot|$ stands for the absolute value of a number, whereas the symbol $\|\cdot\|$, the Euclidean norm of a vector. Denote by $\Xi(x(\cdot), h)$ the set of admissible measurements, i.e., the set of all piecewise constant functions $\xi^h(\cdot) \to R^3$, $\xi^h(t) = \xi_i^h$ for $t \in [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{i,h}$, satisfying inequalities (5). Here

$$\xi_i^h = \{\xi_{1i}^h, \xi_{2i}^h, \xi_{3i}^h\}, \quad \xi^h(t) = \{\xi_{1i}^h, \xi_{2i}^h, \xi_{3i}^h\} \quad \text{for a.a.} \quad t \in [\tau_{i,h}, \tau_{i+1,h}).$$

The control problem under discussion in the paper is as follows. A number $\varepsilon > 0$ is given. It is required to construct an algorithm for forming a feedback control

$$u = u^h(t) = u(t; x^h(\cdot), x_*(\cdot), \xi^h(\cdot))$$

of system (2) providing fulfilment of the following condition. Whatever unknown possible Lebesgue measurable functions $\mu_*(\cdot)$ and $I_*(\cdot)$ with properties (4) may be, the distance between $x^h(t)$ and $x_*(t)$ at all moments $t \in [0, \vartheta]$ should not exceed the value of ε provided the values of h and δ are sufficiently small.

Here

$$x^h(\cdot) = x(\cdot; u^h(\cdot)) = \{T_0^h(\cdot), T_1^h(\cdot), M_1^h(\cdot), K^h(\cdot)\}$$

is the trajectory of Σ generated by the control

$$u(t) = u^h(t; x^h, x_*, \xi^h) = \{\mu^h(t) = \mu(t; x^h(\cdot), x_*(\cdot), \xi^h(\cdot)),\ I^h(t) = I(t; x^h(\cdot), x_*(\cdot), \xi^h(\cdot))\} \in$$
$$\in U(t, x^h(\cdot), x_*(\cdot), \xi^h(\cdot)) \subset [I_-, I_+] \times [f_-, f_+],$$

which is formed according to the feedback principle. Thus, $x^h(\cdot)$ is the solution of system (2) with the feedback controls $\mu(\cdot) = \mu^h(\cdot)$ and $I(\cdot) = I^h(\cdot)$.

Hereinafter, the symbol $\mathscr{U}$ stands for the set of admissible controls, i.e., the set of Lebesgue measurable functions $u(\cdot) = \{\mu(\cdot), I(\cdot)\}$ such that $\mu(t) \in [f_-, f_+]$, $I(t) \in [I_-, I_+]$ for a.a. $t \in [0, \vartheta]$.

One of the approaches to solving the problems of guaranteed control (they are also called positional differential games) for dynamical systems described by ordinary differential equations was developed in [2, 6, 7]. In all the works cited above, the cases when the full phase state of a system is inaccurately measured at frequent enough time moments are considered. In the present work, from the position of the approach described in [2, 6, 7], the problems of guaranteed control under the measurement of a "part" of system's phase state (a "part" of coordinates) are investigated.

To form a control u providing the solution of the problem, along with the information on the "part" of coordinates of the solution of the system Σ (namely, on the values ξ_i^h satisfying inequalities (5)), it is necessary to obtain some additional information on the coordinate $M_1(\cdot)$, which is missing. To get such a piece of information during the control process, it is reasonable, following the approach developed in [2, 6, 7], to introduce an auxiliary controlled system M. This system is described by a differential equation (the form is specified below). The equation has an output $w^h(t)$ and an input $v^h(t)$. The input $v^h(\cdot)$ is some new auxiliary control; it should be formed by the feedback principle in such a way that $v^h(\cdot)$ "approximates" the unknown coordinate $M_1(\cdot)$ in the mean uniform metric. Thus, along with the block of forming the control in the real system (it is called a controller), we need to incorporate into the control contour one more block (it is called a identifier) allowing to reconstruct the missing coordinate $M_1(\cdot)$ in the real time mode.

The scheme of algorithms for solving the problem is given in Figure 1.

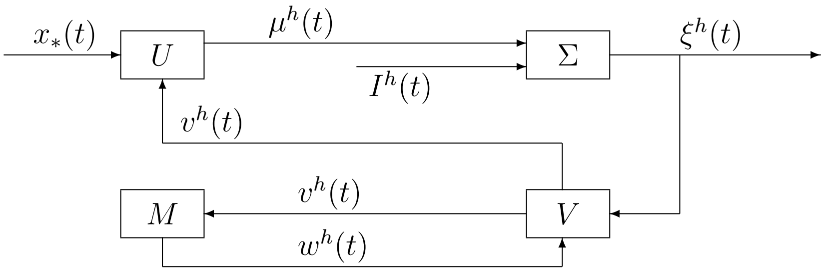

Figure 1.

In the beginning, an auxiliary dynamical system M (a model) is introduced. This model functioning on the time interval $[0, \vartheta]$ has an input $v^h(t)$ and an output $w^h(t)$. The model M with its control law V forms the identifier. Before the algorithm starts, the value h and the partition Δ with the step δ, as well as the model M, are fixed. The process of synchronous feedback control of the systems Σ and M is organized on the interval $[0,\vartheta]$. This process is decomposed into $(m-1)$ identical steps. At the ith step carried out during the time interval $\delta_i = [\tau_i, \tau_{i+1})$, the following actions are fulfilled. First, at the time moment τ_i according to the chosen rules U and V the functions

$$v^h(t) = v_i^h \in V(\tau_i, \xi_i^h, w^h(\tau_i)), \quad t \in \delta_i, \tag{6}$$

$$u^h(t) = u_i^h \in U(\tau_i, v_i^h, \xi_i^h, x_*(\tau_i)), \tag{7}$$

are calculated by measurements ξ_i^h and $w^h(\tau_i)$. Then (till the moment τ_{i+1}) the control $u = u^h(t)$, $\tau_i \le t < \tau_{i+1}$, is fed onto the input of the system Σ and the control $v = v^h(t)$, $\tau_i \le t < \tau_{i+1}$, onto the input of the model M. The values ξ_{i+1}^h and $w^h(\tau_{i+1})$ are the results of the work of the algorithm at the ith step. The procedure stops at the moment ϑ.

Thus, all complexity of solving these problems is reduced to an appropriate choice of a model M and functions U and V.

So, the problem may be formulated as follows. In the sequel, a family of partitions

$$\Delta_h = \{\tau_{i,h}\}_{h=0}^{m_h}, \quad \tau_{i+1,h} = \tau_{i,h} + \delta(h), \quad \tau_{0,h} = 0, \quad \tau_{m_h,h} = \vartheta$$

of the interval $[0,\vartheta]$ is assumed to be fixed.

Problem of robust control. It is required to specify differential equations of the model M

$$\dot{w}^h(t) = f_1(\xi_i^h, w^h(\tau_i), v_i^h), \quad t \in \delta_{h,i} = [\tau_{i,h}, \tau_{i+1,h}), \quad \tau_i = \tau_{i,h}, \tag{8}$$
$$w^h(0) = w_0^h, \quad w^h(t) \in R,$$

and the rule of choosing controls v_i^h and u_i^h at the moments τ_i being a mapping of form (6), (7) such that the inequality

$$\max_{t\in[0,\vartheta]} \|x^h(t) - x_*(t)\| \le \varepsilon \tag{9}$$

holds for $h \in (0, h_*(\varepsilon))$ and $\delta = \delta(h) \in (0, \delta(h_*(\varepsilon))$. Let the symbol $X(\cdot)$ denote the bundle of solutions of system (2), i.e.,

$$X(\cdot) = \{x(\cdot) : x(\cdot) = x(\cdot; x(0), u(\cdot)) = \{T_0(\cdot), T_1(\cdot), M_1(\cdot), K(\cdot)\}, u(\cdot) \in \mathscr{U}\}.$$

We assume that the following condition is fulfilled:

Condition 1.

$$d_* = \inf\Big\{\min_{t\in[0,\vartheta]} \left(1 + \frac{M_1(t)}{590}\right) : x(\cdot) = \{T_0(\cdot), T_1(\cdot), M_1(\cdot), K(\cdot)\} \in X(\cdot)\Big\} > 1.$$

In addition, the functions $\sigma(t)$, $A(t)$, $L(t)$, and $Q(t)$ are considered as known and continuous.

2. Algorithm for reconstructing $M_1(\cdot)$

First, we specify the algorithm for reconstructing $M_1(\cdot)$, which will be applied for solving the problem in question. Namely, we describe the identifier (see Fig. 1). To substantiate this algorithm, we use ideas from [6, 7, 1, 3].

Introduce the notation

$$T(t) = \{T_0(t), T_1(t)\}, \quad f(t, T(t)) = c_1 T_0(t) + c_2 T_1(t) + c_3 Q(t), \quad \tilde{u}(t) = \log_2\left(1 + \frac{M_1(t)}{590}\right).$$

Here $x(\cdot) = \{T_0(\cdot), T_1(\cdot), M_1(\cdot), K(\cdot)\}$ is an arbitrary element of the set $X(\cdot)$. In this case, the first equation of system (2) is rewritten in the form

$$\dot{T}_0(t) = f(t, T(t)) + 4{,}1 c_3 \tilde{u}(t).$$

Note that one can specify a number $M_* > 0$ such that the following inequalities are valid:

$$\|\dot{T}(t)\| \le M_* \quad \text{for almost all} \quad t \in [0, \vartheta], \tag{10}$$

$$\|f(t, T(t)) - f(\tau_i, \xi_i^h)\| \le M_*(\delta + h + \omega(\delta)) \text{ for } t \in \delta_i = [\tau_i, \tau_{i+1}). \tag{11}$$

Here $\tau_i = \tau_{i,h}$, $\omega(\delta)$ is the continuity modulo of the function $t \to O(t)$, $t \in [0, \vartheta]$, i. e.,

$$\omega(\delta) = \sup\{|O(t) - O(t - \delta)| : t \in [\delta, \vartheta]\}.$$

Inequality (11) is a consequence of (5) and (10).

We fix a family Δ_h of partitions of the interval $[0, \vartheta]$ and some auxiliary function $\alpha(h) : (0, 1) \to (0, 1)$.

As the model M, we take a linear system described by a scalar differential equation of the form

$$\dot{w}^h(t) = f(\tau_i, \xi_i^h) + 4{,}1 c_3 v^h(t) \quad \text{for a.a.} \quad t \in \delta_i = [\tau_i, \tau_{i+1}), \tag{12}$$

$i \in [0 : m-1]$, $\quad \tau_i = \tau_{i,h}$, $\quad m = m_h$, with the initial condition

$$w^h(0) = T_0(0).$$

Let

$$v^h(t) = v_i^h \in V(\tau_i, \xi_i^h, w^h(\tau_i)) = -\frac{1}{\alpha} 4{,}1 c_3 [w^h(\tau_i) - \xi_{1i}^h] \quad \text{for} \quad t \in \delta_i. \tag{13}$$

The control $v^h(t)$ in equation (12) is found from (13). Thus, the model control is specified by the feedback principle (see (6)). Consequently, equation (12) takes the form

$$\dot{w}^h(t) = f(\tau_i, \xi_i^h) - \frac{1}{\alpha}(4{,}1 c_3)^2 [w^h(\tau_i) - \xi_{1i}^h] \quad \text{for a.a.} \quad t \in \delta_i. \tag{14}$$

Let us describe the algorithm for reconstructing the unmeasured coordinate $M_1(\cdot)$ in the real time mode. Before the algorithm starts, we fix a value $h \in (0,1)$ and a partition Δ_h. The work of the algorithm is decomposed into $m-1$ identical steps. At the ith step carried out during the time interval $\delta_i = [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{i,h}$, the following actions are fulfilled. First, at the moment τ_i, the control $v^h(t)$ is calculated by (13). This control is fed onto the input of model(12) on the interval $[\tau_i, \tau_{i+1})$. Under the action of this control, the model passes form the state $w^h(\tau_i)$ to the state $w^h(\tau_{i+1}) = w^h(\tau_{i+1}; \tau_i, w^h(\tau_i), v_i^h)$. The work of the algorithm stops at the moment ϑ.

Lemma 1. *Let the conditions*

$$\alpha(h) \to 0,\ \delta(h) \to 0,\ \delta(h)\alpha^{-1}(h) \to 0,\ h\alpha^{-1}(h) \to 0 \text{ as } h \to 0 \tag{15}$$

be fulfilled. Then, uniformly in all $x(\cdot) \in X(\cdot)$, $h \in (0,1)$, $\xi^h(\cdot) \in \Xi(x(\cdot),h)$, $i \in [0 : m_h - 1]$, the inequalities

$$\int\limits_{\tau_i}^{\tau_{i+1}} |\dot{w}^h(s)|\, ds \le C\delta, \tag{16}$$

are valid. Here $C = \mathrm{const} > 0$, $\delta = \delta(h)$, and $\tau_i = \tau_{i,h}$.

Lemma 2. *Let conditions (15) be fulfilled. Let $\delta^\gamma(h)\alpha^{-1}(h) \to +\infty$ (for some $\gamma \in (0,1)$) as $h \to 0$ and*

$$u_e^h(t) = \begin{cases} \tilde{u}(0), & t \in [0, \delta^\gamma) \\ v^h(t), & t \in [\delta^\gamma, \vartheta]. \end{cases}$$

Then, the inequality

$$\sup_{t \in [0,\vartheta]} |u_e^h(t) - \tilde{u}(t)| \le$$

$$\le d_1^0\alpha(h) + d_2^0(h + \delta(h))\alpha^{-1}(h) + d_3^0\omega(\delta(h)) + d_4^0\alpha(h)\delta^{-\gamma}(h) + d_5^0\delta^\gamma(h)$$

is valid. Here the constants d_j^0, $j \in [1 : 5]$, do not depend on $h \in (0,1)$.

Introduce the notation

$$u_*^h(t) = 590(2^{u_e^h(t)} - 1).$$

The following Theorem is true.

Theorem 1. *Under the conditions of Lemma 2, the inequality*

$$\sup_{t \in [0,\vartheta]} |u_*^h(t) - M_1(t)| \le \nu(h, \delta(h), \alpha(h)) =$$

$$= d_1\alpha(h) + d_2(h + \delta(h))\alpha^{-1}(h) + d_3\omega(\delta(h)) + d_4\alpha(h)\delta^{-\gamma}(h) + d_5\delta^\gamma(h)$$

holds. Here the constants d_j, $j \in [1 : 5]$, do not depend on $h \in (0,1)$.

The Theorem follows from Lemma 2 and the inequality

$$|u_*^h(t) - M_1(t)| \le 590|2^{u_e^h(t)} - 2^{\tilde{u}(t)}|.$$

3. Algorithm for solving control problem

Let us turn to the description of the algorithm for solving the control problem in question. From the above, it is necessary to specify model (8) and strategies U and V (6), (7) providing inequality (9).

We fix a family Δ_h of partitions of the interval $[0, \vartheta]$ and some function $\alpha(h) : (0,1) \to (0,1)$. Let the family Δ_h and function $\alpha(h)$ be such that the following condition holds:

Condition 2. *The convergences*

$$\alpha(h) \to 0,\ \delta(h) \to 0,\ \delta(h)\alpha^{-1}(h) \to 0,\ h\alpha^{-1}(h) \to 0, \quad \alpha^{-1}(h)\delta^{\gamma}(h) \to +\infty \quad as \quad h \to 0$$

take place for some $\gamma \in (0,1)$.

Let model (8) be of form (12), i.e.,

$$\dot{w}^h(t) = f(\tau_i, \xi_i^h) + 4,1c_3 v^h(t) \quad \text{for a.a.} \quad t \in \delta_i = [\tau_i, \tau_{i+1}), \tag{17}$$

$i \in [0 : m-1], \quad \tau_i = \tau_{i,h}, \quad m = m_h$, with the initial condition

$$w^h(0) = T_0(0).$$

Let rules U (6) and V (7) for forming the controls u_i^h and v_i^h be as follows:

$$v_i^h = V(\tau_i, \xi_i^h, w^h(\tau_i)) = -\frac{1}{\alpha}4,1c_3[w^h(\tau_i) - \xi_{1i}^h], \tag{18}$$

$$u_i^h = \{\mu^h(\tau_i), I^h(\tau_i)\} = U(\tau_i, v_i^h, \xi_i^h, x_*(\tau_i)) \quad \text{for} \quad t \in \delta_i.$$

Here

$$I^h(\tau_i) = \arg\min\{(\xi_{3i}^h - K_*(\tau_i))I : I \in [I_-, I_+]\}, \tag{19}$$

$$\mu^h(\tau_i) = \arg\min\{E_1(\tau_i, K_*)(u_*^h(\tau_i) - M_{1*}(\tau_i))\mu : \mu \in [f_-, f_+]\}, \tag{20}$$

$$u_*^h(\tau_i) = \begin{cases} \log_2\Big(1 + \frac{M_{*1}(t)}{590}\Big), & \text{if } \tau_i \le \delta^{\gamma}(h) \\ v_i^h, & \text{otherwise.} \end{cases}$$

In what follows, we need

Lemma 3. [4] *Let the function $\varepsilon(t)$ be nonpositive for $t \in T$ and, for all $i \in [0 : m-1]$, satisfy the inequalities*

$$\varepsilon(\tau_{i+1}) \leq \varepsilon(\tau_i)(1+\beta\delta) + \int\limits_{\tau_i}^{\tau_{i+1}} |\varphi(t)|\,dt,$$

where $\tau_i \in \Delta$, $\beta = \mathrm{const} > 0$, and $\varphi(\cdot) \in L(T; R)$. Then,

$$\varepsilon(\tau_i) \leq \Big(\varepsilon(t_0) + \int\limits_{t_0}^{\tau_i} |\varphi(t)|\,dt\Big) \exp(\beta(\tau_i - t_0)).$$

Introduce

Condition 3. *The inequalities*

$$0 < C^{(1)} < K_*(t) < C^{(2)} < +\infty \quad \textit{for } t \in [0, \vartheta]$$

are valid.

Theorem 2. *For any $\varepsilon > 0$, one can specify $h_*(\varepsilon) \in (0,1)$ such that, for all $h \in (0, h_*(\varepsilon))$ and $\delta(h) \in (0, \delta(h_*(\varepsilon)))$, inequality (9) holds, if the model M is given by equation (17), the strategies V and U are taken in form* (6), (7), (18)*–(20).*

The proof of the theorem is performed by the scheme of the proof of corresponding statements from [2] and is based on Theorem 3 and Lemma 4. In the process, the variation of the values

$$\varepsilon_1(t) = |K_*(t) - K^h(t)|^2, \quad t \in [0, \vartheta],$$
$$\varepsilon_2(t) = |M_{*1}(t) - M_1^h(t)|^2, \quad t \in [0, \vartheta]$$

is estimated and the inequalities

$$\varepsilon_1(t) \leq C_*(\delta + h),$$
$$\varepsilon_2(t) \leq C_{**}(h^{1/2} + \delta^{1/2} + \nu(h, \delta(h), \alpha(h))), \quad t \in [0, \vartheta]$$

are established. Here the function $\nu(h, \delta, \alpha)$ is defined in Theorem 3.

In the conclusion, we describe the algorithm of the problem under consideration. Thus, we have system (2) with the control $u = \{\mu, I\}$ and system (3) with the unknown control $u_* = \{\mu_*, I_*\}$. We choose a family $\Delta_h = \{\tau_{i,h}\}_{i=0}^{m_h}$ of partitions of the interval $[0, \vartheta]$ with a step $\delta(h) = \tau_{i+1,h} - \tau_{i,h}$ and a function $\alpha(h) : (0,1) \to (0,1)$ depending on the parameter h. The family Δ_h and function $\alpha(h)$ satisfy Condition 2. Before the algorithm starts, we fix some value of measurement accuracy h, the partition $\Delta = \Delta_h$ and number $\alpha = \alpha(h)$. The work of the algorithm is decomposed into $m-1$, $m = m_h$, identical steps.

At the ith step carried out during the time interval $\delta_i = [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{i,h}$, the following actions are fulfilled. First, at the moment τ_i, using the state $w^h(\tau_i)$ of model (17), the result ξ_i^h (satisfying inequality (5)) of calculating the state of system (2), we calculate three numbers, namely, v_i^h and $u_i^h = \{\mu^h(\tau_i), I^h(\tau_i)\}$, by formulas (18)–(20). Then, during the time interval δ_i, the constant control $u^h(t) = u_i^h$ is fed onto the input of model (17). After these operations, at the moment τ_{i+1} the model state is recalculated (instead of the number $w^h(\tau_i)$, the number $w^h(\tau_{i+1}) = w^h(\tau_{i+1}; w^h(\tau_i), v_i^h)$ is found; in addition, the vector ξ_{i+1}^h is determined). The analogous actions are performed till the moment $\tau_{m_h-1,h}$.

As follows from Theorem 1, if the fixed measurement accuracy h is sufficiently small, then the described above algorithm for forming the control $u(\cdot)$ in system (2) provides "tracking" (in uniform metric) the solution $x_*(\cdot)$ of system (3) by the solution $x^h(\cdot)$ of system (2). Thus, the algorithm solves the problem of robust control.

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