AVERAGING IN THE OPTIMAL CONTROL PROBLEM FOR THE REACTION-DIFFUSION EQUATION WITH MULTIVALUED INTERACTION FUNCTION

© Olena Kapustian

Taras Shevchenko National University of Kyiv
Faculty of Cybernetics
e-mail: *olena.kap@gmail.com*

Abstract. *In this paper consider the optimal control problem on infinite time interval with quadratic cost functional. State of this problem is defined by the evolutionary inclusion of reaction-diffusion type. We prove the solvability of such a problem. In the case of rapidly oscillating coefficients in coefficients of differential operator and multivalued interaction function we prove the convergence of* ε*-dependent optimal process to optimal process of the corresponding averaged problem.*

Introduction

One of the main problems in the study of processes in micro-inhomogeneous media is the correctness of passing to the averaged problem [1]. Works [2] - [4] are devoted to the research on convergence in optimal control problems for distributed systems with perturbations in coefficients. In this paper we consider the optimal control problem on the solutions of reaction-diffusion type inclusion. Moreover, such an inclusion has perturbations in the differential operator coefficients and multivalued interaction function which has power growth. We investigate the issue of the solution dependence on the parameter for mentioned problem. However as opposed to [3, 4] the averaged problem is not degenerate into linear-quadratic one.

1. Problem setting

We consider the optimal control problem

$$\begin{cases} \frac{\partial y}{\partial t} \in div(a^{\varepsilon}(x)\nabla y) - F_{\varepsilon}(x,y) + h^{\varepsilon}(x)u(t), \ x \in \Omega, \ t > 0, \\ y(x,t) = 0, \ x \in \partial\Omega, \\ y(x,0) = y_0^{\varepsilon}, \end{cases} \tag{1}$$

$$u(t) \in U \subseteq L^2(0,+\infty), \tag{2}$$

$$J(y,u) = \int\limits_0^{+\infty} \int\limits_{\Omega} y^2(x,t)dxdt + \gamma \int\limits_0^{+\infty} u^2(t)dt \to \inf, \tag{3}$$

where $\Omega \subset \mathbb{R}^n$ is bounded domain, $\varepsilon > 0$ is a small parameter, matrix $a^\varepsilon(x) = \{a^\varepsilon_{ij}(x)\}$ is measurable, symmetric and satisfies the condition of uniform ellipticity

$$\begin{gathered} \exists \lambda_1 > 0,\ \Lambda_1 > 0\ \forall \varepsilon > 0\ \forall \xi \in \mathbb{R}^n \\ \lambda_1 |\xi|^2 \leq \sum_{i,j=1}^{n} a^\varepsilon_{ij}(x)\xi_i\xi_j \leq \Lambda_1 |\xi|^2. \end{gathered} \tag{4}$$

Multivalued interaction function $F_\varepsilon(x, y)$ has a form

$$F_\varepsilon(x, y) = [b_\varepsilon(x) f_\varepsilon(y), d_\varepsilon(x) g_\varepsilon(y)].$$

Here $b_\varepsilon,\ d_\varepsilon$ are measurable, bounded functions in $L^\infty(\Omega)$, for which the following condition holds

$$\exists \beta > 0\ \forall x \in \Omega\ \forall\ \varepsilon > 0\ b_\varepsilon(x) \geq \beta,\ d_\varepsilon(x) \geq \beta. \tag{5}$$

Functions $f_\varepsilon,\ g_\varepsilon$ are bounded functions in $C(\mathbb{R})$, which satisfy the next conditions

$$\begin{gathered} \exists C_1 \geq 0,\ \exists \alpha > 0,\ \exists p \geq 0,\ \forall y \in \mathbb{R}\ \forall \varepsilon > 0 \\ |f_\varepsilon(y)| + |g_\varepsilon(y)| \leq C_1(1 + |y|^{p-1}), \\ y f_\varepsilon(y) \geq \alpha |y|^p, y g_\varepsilon(y) \geq \alpha |y|^p. \end{gathered} \tag{6}$$

Functions $h^\varepsilon,\ y^\varepsilon_0$ are bounded in $L^2(\Omega)$, set of admissible controls U is closed, convex and $0 \in U$.

Definition. For fixed $u \in U$ a function $y \in W = L^2_{loc}(0, +\infty; H^1_0(\Omega)) \bigcap L^p_{loc}(0, +\infty; L^p(\Omega))$ is called the solution of the problem (1) if this function is such that $y(0) = y^\varepsilon_0$, and for some function $l = l(t, x) \in L^q_{loc}(0, +\infty; L^q(\Omega)),\ \frac{1}{p} + \frac{1}{q} = 1$ it holds that $l(t, x) \in F_\varepsilon(x, y(t, x))$ almost everywhere (a. e.) and $\forall v \in H^1_0(\Omega) \bigcap L^p(\Omega)$, $\forall \eta \in C^\infty_0(0, T)$

$$\int_0^T (y, v)\eta_t dt - \int_0^T ((a^\varepsilon \nabla y, \nabla y) + (l, v) - u(t)(h^\varepsilon, v))\, \eta dt = 0. \tag{7}$$

Here and below $\|\cdot\|$ and $(\cdot, \cdot)$ indicate a norm and a scalar product in $L^2(\Omega)$.

By the conditions (5), (6) the global solvability of the problem (1) follows from [5] for $\forall u \in U$, $y^\varepsilon_0 \in L^2(\Omega)$, if in the right-hand side we put a continuous selector of the mapping F_ε. However from the results of [6] it implies that the set of the solutions of (1) is not exhausted to the solutions of equations for continuous selectors of F_ε. It greatly increases the set of admissible processes in the problem (1) - (3).

The main aim of this paper is to prove convergence of optimal process of the problem (1) - (3) to optimal process of corresponding averaged problem.

2. Existence of solutions of the optimal control problem

From [3] - [6] it follows that any solution of the problem (1) belongs to class $C([0,+\infty);L^2(\Omega))$ and for almost all (a. a.) $t>0$ next energy equality holds

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|^2+(a^\varepsilon\nabla y(t),\nabla y(t))+(l,y(t))=u(t)(h^\varepsilon,y(t)), \tag{8}$$

where $l(t,x)\in F_\varepsilon(x,y(t,x))$ a. e.

Moreover, by (4) - (6) $\forall t\geq s\geq 0$ we have

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|^2+\lambda_1\|y(t)\|^2_{H_0^1}+\alpha\beta\|y(t)\|^p_{L^p}\leq|u(t)|\|h^\varepsilon\|\|y(t)\|. \tag{9}$$

From the Poincare inequality [5] one can obtain

$$\frac{d}{dt}\|y(t)\|^2+\lambda_1\|y(t)\|^2_{H_0^1}+2\alpha\beta\|y(t)\|^p_{L^p}\leq C_2\|h^\varepsilon\|^2|u(t)|^2. \tag{10}$$

Applying the Gronwall inequality, we finally have $\forall t\geq s\geq 0$

$$\|y(t)\|^2\leq\|y(s)\|^2\exp^{-\lambda_1(t-s)}+C_3|h^\varepsilon\|^2\int\limits_0^{+\infty}|u(t)|^2dt. \tag{11}$$

Using the Poincare inequality again, by (10) $\forall t\geq s\geq 0$ we have

$$\int\limits_s^t\|y(s)\|^2ds\leq\frac{1}{\lambda_1}\big(\|y(t)\|^2+\|y(s)\|^2+C_2\|h_\varepsilon\|^2\int\limits_s^t|u(s)|^2ds\big). \tag{12}$$

Herefrom, in particular, this implies that $J(y,u)<\infty$.

The next lemma is needed for passing to the limit in the problem (1) and it follows from The Mazur Theorem [7].

Lemma 1. *Let Q be a bounded set, $q\geq 1$ and functions f_n, q_n, $l_n\in L^q(Q)$ satisfy*

$$f_n(x)\leq l_n(x)\leq g_n(x)\ for\ \ a.\ a.\ x\in Q,$$

$$f_n\to f,\ l_n\to l,\ g_n\to g\ weakly\ \ in\ L^q(Q).$$

Then

$$f(x)\leq l(x)\leq g(x)\ for\ \ a.\ a.\ x\in Q.$$

Theorem 1. *Under (4) - (6) for $\forall\varepsilon>0$, $\forall y_0^\varepsilon\in L^2(\Omega)$ the optimal control problem (1) - (3) has at least one solution.*

Доказательство. Let $\widetilde{J}_\varepsilon$ be a value of the problem (1) - (3). We choose $\{u_n\}\subset U$ such that $\forall n\geq 1$

$$J(y_n,u_n)\leq\widetilde{J}_\varepsilon+\frac{1}{n}.$$

Then

$$\gamma \int\limits_0^{+\infty} |u_n(t)|^2 dt \le \widetilde{J}_\varepsilon + \frac{1}{n},$$

so $\{u_n\}$ is bounded in $L^2(0,+\infty)$ and for some $u \in U$ on subsequence

$$u_n \to u \text{ weakly in } L^2(0,+\infty).$$

From the estimates (10), (11) for $\forall T > 0$

$$\{y_n\} \text{ is bounded in } L^\infty(0,T;L^2(\Omega)) \bigcap L^2(0,T;H_0^1(\Omega)) \bigcap L^p(0,T;L^p(\Omega)). \tag{13}$$

By the condition (6) we have

$$\{f_\varepsilon(y_n)\}, \{g_\varepsilon(y_n)\} \text{ are bounded in } L^q(0,T;L^q(\Omega)).$$

For $l_n(t,x) \in F_\varepsilon(x, y_n(t,x))$ $\exists \lambda_n = \lambda_n(t,x) \in [0,1]$ such that for a. a. (t,x)

$$l_n(t,x) = \lambda_n b_\varepsilon(x) f_\varepsilon(y_n(t,x)) + (1-\lambda_n) d_\varepsilon(x) g_\varepsilon(y_n(t,x)).$$

And since b_ε, d_ε are bounded in $L^\infty(\Omega)$, then

$$\{l_n\} \text{ is bounded in } L^q(0,T;L^q(\Omega)). \tag{14}$$

This implies that

$$\{\frac{\partial y_n}{\partial t}\} \text{ is bounded in } L^2(0,T;H^{-1}(\Omega)) + L^q(0,T;L^q(\Omega)). \tag{15}$$

From the Compactness Theorem [5] for some function $y \in W$ on subsequence

$$\begin{gathered} y_n \xrightarrow{w} y \text{ in } L^2(0,T;H_0^1(\Omega)), \\ y_n \to y \text{ in } L^2(0,T;L^2(\Omega)), \\ y_n(t) \xrightarrow{w} y(t) \text{ in } L^2(\Omega)\ \forall\, t \ge 0, \\ y_n(t) \to y(t) \text{ in } L^2(\Omega) \text{ for a.a. } t \ge 0, \\ y_n(t,x) \to y(t,x) \text{ a. e.}, \\ l_n \xrightarrow{w} l \text{ in } L^q(0,T;L^q(\Omega)). \end{gathered} \tag{16}$$

Passing to the limit in (7) at $n \to \infty$, we have that $\{y, u, l\}$ satisfies (7).

By Lions Lemma [5] $b_\varepsilon f_\varepsilon(y_n) \to b_\varepsilon f_\varepsilon(y)$, $d_\varepsilon g_\varepsilon(y_n) \to d_\varepsilon g_\varepsilon(y)$ at $n \to \infty$ weakly in $L^q((0,T)\times\Omega)$ and a.e. In this case for a. a. (t,x)

$$b_\varepsilon f_\varepsilon(y_n(t,x)) \le l_n(t,x) \le d_\varepsilon g_\varepsilon(y_n(t,x)).$$

Then from the Lemma 1 $l(t,x) \in F_\varepsilon(x, y(t,x))$ a. e.

Hence, $\{y,u\}$ is the admissible process in the problem (1) - (3), and inequality

$$J(y_n,u_n) \geq J_T(y_n,u_n) := \int_0^T \|y_n(t)\|^2 dt + \gamma \int_0^T |u_n(t)|^2 dt$$

implies that $\forall T > 0$

$$\widetilde{J}_\varepsilon \geq \varliminf_{n\to\infty} J(y_n,u_n) \geq \varliminf_{n\to\infty} J_T(y_n,u_n) \geq$$
$$\geq \varliminf_{n\to\infty} \int_0^T \|y_n(t)\|^2 dt + \gamma \varliminf_{n\to\infty} \int_0^T |u_n(t)|^2 dt \geq J_T(y,u).$$

It follows that $\widetilde{J}_\varepsilon = J(y,u)$, so $\{y,u\}$ is the optimal process of the problem (1) - (3).

□

3. Convergence to optimal process of averaged problem

Let us consider now a limit averaged problem

$$\begin{cases} \frac{\partial y}{\partial t} = div(a^0(x)\nabla y) - F_0(x,y) + h_0(x)u(t),\ x \in \Omega,\ t > 0, \\ y(x,t) = 0,\ x \in \partial\Omega, \\ y(x,0) = y_0, \end{cases} \tag{17}$$

$$u(t) \in U \subseteq L^2(0,+\infty), \tag{18}$$

$$J(y,u) = \int_0^{+\infty} \int_\Omega y^2(x,t)dxdt + \gamma \int_0^{+\infty} u^2(t)dt \to \inf, \tag{19}$$

where $F_0(x,y) = b(x)f(y)$ and for $\varepsilon \to 0$

$$\begin{gathered} a^\varepsilon \to a^0,\ h^\varepsilon \to h_0 \text{ in } L^2(\Omega), \\ y_0^\varepsilon \to y_0 \text{ weakly in } L^2(\Omega), \\ b_\varepsilon \to b,\ d_\varepsilon \to b \text{ *-weakly in } L^\infty(\Omega), \\ \forall R > 0 \sup_{|y|\leq R} (|f_\varepsilon(y) - f(y)| + |g_\varepsilon(y) - f(y)|) \to 0. \end{gathered} \tag{20}$$

By (20) this implies that the matrix $a(x)$ is symmetric and satisfies (4), $b(x)$ satisfies (5) and $f \in C(\mathbb{R})$ satisfies (6). Hence, by Theorem 1 the optimal control problem (17) - (19) has solutions and we can consider the problem (1) as the perturbed problem (17). Such a situation naturally arises when modeling of complex evolutionary processes in micro-inhomogeneous media.

The following condition is supposed to satisfy:

$$\forall u \in U\ \forall y_0 \in L^2(\Omega) \text{ the problem (17) has the unique solution.} \tag{21}$$

The following condition [5] is sufficient to carry out the condition (21):

$$f \in C^1(\mathbb{R}),\ f'(u) \geq -C_4\ \forall u \in \mathbb{R}.$$

Theorem 2. *Let the conditions (4) - (6), (20), (21) hold. Then*

$$\lim_{\varepsilon\to 0} |\widetilde{J}_\varepsilon - \widetilde{J}_0| = 0,$$

where $\widetilde{J}_\varepsilon$ is the value of the problem (1) - (3), $\widetilde{J}_0$ is the value of the problem (17) - (19).

Доказательство. Let $\{\widetilde{y}^\varepsilon, \widetilde{u}^\varepsilon\}$ be an optimal process of the problem (1) - (3). Note that for any admissible process $\{y, u\}$ in the problem (1) - (3) the estimates (10), (11) are valid. Therefore if z^ε is the solution of (1) with control $u \equiv 0 \in U$, then by the optimality of $\widetilde{u}^\varepsilon$ we have

$$\int_0^{+\infty} |\widetilde{u}^\varepsilon(t)|^2 dt \leq \frac{1}{\gamma}\int_0^{+\infty} \|z^\varepsilon(t)\|^2 dt \leq \frac{1}{\gamma}\int_0^{+\infty} \|y_0^\varepsilon\|^2 e^{-\lambda_1 t} dt \leq \frac{\|y_0^\varepsilon\|^2}{\lambda_1\gamma}. \tag{22}$$

Hence $\{\widetilde{u}^\varepsilon\}$ is bounded in $L^2(0,+\infty)$ and for some $\widetilde{u} \in U$ on subsequence

$$\widetilde{u}^\varepsilon \to \widetilde{u} \text{ weakly in } L^2(0,+\infty).$$

Let $\widetilde{l}^\varepsilon$ corresponds to $\widetilde{y}^\varepsilon$, $\widetilde{l}^\varepsilon(t,x) \in F_\varepsilon(x, \widetilde{y}^\varepsilon(t,x))$ a. e. Then we can repeat thinking of the Theorem 1 and obtain the convergence (16) for some $\widetilde{y} \in W$, $\widetilde{l} \in L^q((0,T)\times\Omega)$.

Let's argue the passing to the limit in the equality (7). Since $a^\varepsilon \to a^0$ in $L^2(\Omega)$ then

$$\int_0^T (a^\varepsilon \nabla \widetilde{y}^\varepsilon, \nabla v)\eta dt \to \int_0^T (a\nabla\widetilde{y}, \nabla v)\eta dt\ \forall v \in H_0^1(\Omega), \forall \eta \in C_0^\infty(0,T).$$

Due to strong convergence $a^\varepsilon \to a^0$, $h^\varepsilon \to h_0$ in $L^2(\Omega)$, we can pass to the limit in the equality (7) and obtain that $\{\widetilde{y}, \widetilde{u}, \widetilde{l}\}$ satisfies (7) for $\forall T > 0$.

Prove that $\widetilde{l}(t,x) = b(x)f(\widetilde{y}(t,x))$ a. e. In fact, $f_\varepsilon(\widetilde{y}^\varepsilon) \to f(\widetilde{y})$ weakly in $L^q(0,T;L^q(\Omega))$ and a. e., $b_\varepsilon \to b$ *-weakly in $L^\infty(\Omega)$. Then

$$b_\varepsilon f_\varepsilon(\widetilde{y}^\varepsilon) - bf(\widetilde{y}) = b_\varepsilon(f_\varepsilon(\widetilde{y}^\varepsilon) - f(\widetilde{y})) + (b_\varepsilon - b)f(\widetilde{y}) = I_\varepsilon^{(1)}(t,x) + I_\varepsilon^{(2)}(t,x).$$

Since b_ε is bounded in $L^\infty(\Omega)$, then $I_\varepsilon^{(1)}(t,x) \to 0$ a.e. and it is bounded in $L^q(0,T;L^q(\Omega))$. Hence, by Lions Lemma $I_\varepsilon^{(1)}(t,x) \to 0$ weakly in $L^q(0,T;L^q(\Omega))$.

On the other hand, $\forall\theta \in L^p(0,T;L^p(\Omega))$ $f(\widetilde{y})\cdot\theta \in L^1((0,T)\times\Omega)$, therefore

$$\int_0^T\int_\Omega (b_\varepsilon(x) - b(x))f(\widetilde{y}(t,x))\theta(t,x) \to 0,$$

i. e. $I_\varepsilon^{(2)}(t,x) \to 0$ weakly in $L^q(0,T;L^q(\Omega))$.

Thus,

$$b_\varepsilon(x)f_\varepsilon(\widetilde{y}(t,x)) \le \widetilde{l}(t,x) \le d_\varepsilon(x)g_\varepsilon(\widetilde{y}(t,x)) \text{ a. e.},$$

moreover,

$$b_\varepsilon \cdot f_\varepsilon(\widetilde{y}^\varepsilon) \to b\cdot f(\widetilde{y}),\ d_\varepsilon g_\varepsilon(\widetilde{y}) \to b\cdot f(\widetilde{y}) \text{ weakly in } L^q(0,T;L^q(\Omega)),$$

$$\widetilde{l}^\varepsilon \to \widetilde{l} \text{ weakly in } L^q(0,T;L^q(\Omega)).$$

Then by the Lemma 1 we have that $\widetilde{l}(t,x) = b(x)f(\widetilde{y}(t,x))$ a. e.

Moreover, $\widetilde{y}^\varepsilon \to \widetilde{y}$ in $C([\tau,T];L^2(\Omega))\ \ \forall\tau>0$. So $\forall T>0$

$$\underline{\lim}_{\varepsilon\to 0} \widetilde{J}_\varepsilon \ge \underline{\lim}_{\varepsilon\to 0} J_T(\widetilde{y}^\varepsilon,\widetilde{u}^\varepsilon) \ge J_T(\widetilde{y},\widetilde{u}),$$

hence

$$\underline{\lim}_{\varepsilon\to 0} \widetilde{J}_\varepsilon \ge J(\widetilde{y},\widetilde{u}). \tag{23}$$

Using Bellman optimality principle, we can argue [4] that $\{\widetilde{y},\widetilde{u}\}$ is an optimal process of the problem (17) - (19).

Let's prove that

$$\overline{\lim}_{\varepsilon\to 0} \widetilde{J}_\varepsilon \le J(\widetilde{y},\widetilde{u}). \tag{24}$$

From Bellman optimality principal we obtain that the process $\{\widetilde{y}^\varepsilon,\widetilde{u}^\varepsilon\}$ is optimal for the problem (1) – (3) on $[T,+\infty)$ with initial data $(T,\widetilde{y}^\varepsilon(T))$. Then for every $T>0$ by (12) the following inequality holds

$$\int_T^{+\infty} \|\widetilde{y}^\varepsilon(t)\|^2dt + \gamma\int_T^{+\infty} |\widetilde{u}^\varepsilon(t)|^2dt \le \int_T^{+\infty} \|p^\varepsilon(t)\|^2dt \le \frac{1}{\lambda_1}\|\widetilde{y}^\varepsilon(T)\|^2, \tag{25}$$

where p^ε is the solution of the problem (1) with control $u=0\in U$ and initial data $(T,\widetilde{y}^\varepsilon(T))$.

Let ω^ε be a solution of the problem (1) with control $\widetilde{u}$. Then from (21) we have that $\omega^\varepsilon\to\widetilde{y}$ in the sense of (16). Moreover, we obtain the following estimates:

$$\begin{gathered}\int_0^{T} \|\widetilde{y}^\varepsilon(t)\|^2dt + \gamma\int_0^{+\infty} |\widetilde{u}^\varepsilon(t)|^2dt \le \\ \le \gamma\int_0^{+\infty}|\widetilde{u}(t)|^2dt + \int_0^T\|\omega^\varepsilon(t)\|^2dt + \int_T^{+\infty}\|\omega^\varepsilon(t)\|^2dt \le \\ \le \frac{1}{\lambda_1}\|\omega^\varepsilon(T)\|^2 + \gamma\int_0^{+\infty}|\widetilde{u}(t)|^2dt + \int_0^T\|\omega^\varepsilon(t)\|^2dt + \frac{C_1}{\lambda_1}\int_T^{+\infty}|\widetilde{u}(t)|^2dt.\end{gathered} \tag{26}$$

Then

$$\gamma \varlimsup_{\varepsilon\to 0} \int_0^{+\infty} |\widetilde{u}^\varepsilon(t)|^2 dt \le \int_0^{+\infty} |\widetilde{u}(t)|^2 dt + \frac{2}{\lambda_1}\|\widetilde{y}(T)\|^2 + \frac{C_1}{\lambda_1}\int_T^{+\infty} |\widetilde{u}(t)|^2 dt$$

and for $T \to \infty$ we get

$$\varlimsup_{\varepsilon\to 0} \int_0^{+\infty} |\widetilde{u}^\varepsilon(t)|^2 dt \le \int_0^{+\infty} |\widetilde{u}(t)|^2 dt, \tag{27}$$

which together with weak convergence guarantees strong convergence $\widetilde{u}^\varepsilon \to \widetilde{u}$ in $L^2(0,+\infty)$.

Further from inequalities (25), (26) we obtain the following inequality

$$\widetilde{J}_\varepsilon \le J_T(\widetilde{u}^\varepsilon) + \frac{1}{\lambda_1}|\widetilde{y}^\varepsilon(T)|^2.$$

Then

$$\varlimsup_{\varepsilon\to 0} \widetilde{J}_\varepsilon \le J_T(\widetilde{u}) + \frac{1}{\lambda_1}|\widetilde{y}(T)|^2$$

and for $T \to \infty$ we get (24), which means together with (23) that on some subsequence

$$\lim_{\varepsilon\to 0} \widetilde{J}_\varepsilon = J(\widetilde{y}, \widetilde{u}).$$

Assuming by contradiction that this convergence goes on not all sequence $\varepsilon \to 0$, we can repeat previous thinking and under uniqueness of optimal process $\{\widetilde{y}, \widetilde{u}\}$ we obtain the contradiction.

$\square$

Conclusion

In this paper the following results were obtained:

– we proved the solvability of optimal control problem (1)–(3),

– we proved convergence of the optimal process of the problem (1)–(3) to optimal process of corresponding averaged problem (17)–(19).

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