# INCOMPLETENESS OF INITIAL INFORMATION AND THE PROBLEM OF PAYOFF FUNCTION RECONSTRUCTION

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***Abstract.*** *The paper introduces the classification of informational situations for a zero-sum game with incomplete information based on uncertainty level. For each case the possible ways to deal with uncertainty are considered.*

Zero-sum matrix game (antagonistic game) is one of the most popular game-theoretical models widely used in theory and practice [1, 2].

Game-theoretical modeling usually assumes the knowledge of all game components such as the set of all pure strategies and values for all entries of the game payoff matrix. However in practice not all values of payoff matrix elements for the antagonistic game which simulate a decision-making problem are possible to know. This prevents a wide use of game-theoretical models in decision support systems (DSS).

We will call a Partially Defined Antagonistic Game the following generalized form of two-person zero-sum game

1. Given is the set $X = \{1, 2, \ldots, m\}$ of all pure strategies of player I numbered with natural numbers $1, 2, \ldots, m$;
2. Given is the set $Y = \{1, 2, \ldots, n\}$ of all pure strategies of player II numbered with natural numbers $1, 2, \ldots, n$;
3. The payoff matrix $A = A_{m\times n} = (a_{ij})$ is given partially (values for some payoff matrix entries $a_{ij}$ are omitted).

Similar to the classical antagonistic game, $A_{ij}$ represents the winnings of player I when player I chooses pure strategy $i$ and player II chooses pure strategy $j$. The winnings of player I are equal to the loses of player II.

For a partially defined antagonistic game there are numbers $i_0 \in X$ and $j_0 \in Y$ such that the values of correspondent payoff matrix entries $A = A_{m\times n} = (a_{ij})$ are unknown. In the general case a payoff matrix may contain a lot of such elements.

Up to now the problem of getting solutions for the partially defined antagonistic games is described very scarcely in the modern scientific literature.

Different concepts of decision search for partially defined antagonistic games are possible under conditions of risk and uncertainty. The natural way of solution search for a partially defined antagonistic game lies in its correct reduction to some classical antagonistic game. The solution of such the antagonistic game with completely defined payoff matrix can be treated as an optimal solution of the initial partially defined antagonistic game.

Decision-making game model is given by a triplet $< X, Y, R >$, where $X = \{1, 2, \ldots, m\}$ is a set of pure strategies for player I, $Y = \{1, 2, \ldots, n\}$ is a set of pure strategies for player II, $A = A_{m\times n} = (a_{ij})$ is a partially defined payoff matrix of the antagonistic game. There is at least one or several elements $a_{ij}$ with unknown values. Our task is to find optimal strategies (possibly mixed) for players in a partially defined antagonistic game.

Classification of possible informational situations is given below. It is similar to the classification of informational situations given in [3, 4] where a comparison criteria is based on uncertainty level the Nature player encounters while choosing a possible state.

We shall call an Informational Situation (IS) the gradation level characterizing the uncertainty of elements $a_{ij}$ from a partially defined payoff matrix $A = A_{m\times n} = (a_{ij})$.

Informational situation classification can be represented by the following gradation

1. $I_1$ — the first IS: unknown elements of payoff matrix are all random values described by a known distribution;
2. $I_2$ — the second IS: all unknown elements of payoff matrix are represented by functions of one or several parameters;
3. $I_3$ — the third IS: all unknown elements of payoff matrix are restricted by a range of values;
4. $I_4$ — the forth IS: there is no any mathematical information about unknown elements of payoff matrix;
5. $I_5$ — the fifth IS: all unknown elements of payoff matrix takes the worst values for player I that is values preventing player I from reaching his/her aims;
6. $I_6$ — the sixth IS: all unknown elements of payoff matrix belong to a given fuzzy set [4], these elements are represented by fuzzy variables with known membership functions;
7. $I_7$ — the seventh IS: IS intermediate between $I_1$ and $I_6$.

Lets note the particular quality of $I_4$. The situation when all the elements $a_{ij}$ of payoff matrix $A = A_{m\times n} = (a_{ij})$ are unknown is forbidden only for informational situation $I_4$. Indeed, if a payoff matrix is completely unknown in situation $I_4$ then the formalization of completely undefined zero-sum game loses any mathematical meaning.

In virtually all cases of informational situations $I_l$ it is possible to evaluate unknown elements of the payoff matrix by interpolating (or extrapolating) corresponding functions or by using pattern recognition methods.

Consider some possible ways to deal with uncertainty.

In case $I_1$ all unknown elements of payoff matrix are all random values given by a distribution law. In this case it is reasonable to change all elements of the payoff matrix (which are the given random variables) with values of numerical characteristics of the corresponding probability distribution such as mathematical expectations, modal values, as well as variances, standard deviations, coefficients of variance, and other numerical characteristics of these random variables.

In case $I_2$ all unknown elements of payoff matrix are represented by given functions of one or several parameters. One approach to solving the partially defined antagonistic game for this case is based on investigating the effect of possible values of these parameters on the optimal solution of the corresponding game. For some cases this investigation will lead to consideration of analytic (functional) dependencies of the optimal solution. In other cases it will be based on the search over the finite set of the most typical (or most important) parameter values. Moreover, it is possible that the mathematical idea behind the partially defined antagonistic game under consideration requires either a single optimal solution, or a number of optimal solutions which are equivalent with respect to the chosen decision criterion. In this case the final choice of optimal solution may require other approaches (e.g. the operation research methods or the methods of expected utility theory).

In case $I_3$ all unknown entries of payoff matrix are restricted by a range of values. For example, the range of unknown elements can be defined by minimal and maximal values with the inequalities of the form $r^{min}_{i_0 j_0} \le a_{i_0 j_0} \le r^{max}_{i_0 j_0}$. Here $a_{i_0 j_0}$ is a payoff matrix element with unknown true value, $r^{min}_{i_0 j_0}, r^{max}_{i_0 j_0}$ are given numbers satisfying the strict inequality $r^{min}_{i_0 j_0} < r^{max}_{i_0 j_0}$. In such cases one can try an approach based on search among the most typical (and/or most important) values of the corresponding elements of the payoff matrix true values of which are unknown but should meet given restrictions. Though this approach entails a considerable increase in computational operations needed to solve a number of zero-sum games with completely defined payoff matrices.

In case $I_4$ we have only some elements of the payoff matrix. This enables us to say that we have an initial information (a learning data set) which can be used for restoration the unknown elements by the method of empirical generalization. In this case the initial information is treated as a training set containing all necessary information about the matrix. Assuming that there is a regularity (payoff function $H : X \times Y \to \mathbb{R}$) exhibited by the training set we can tackle the problem of function restoration which is incorrect in general case $H$.

In case $I_5$ all unknown elements of payoff matrix takes the values preventing player I (Decision Maker, DM) from reaching his/her aims. Here the economical or physical meaning of the payoff matrix elements plays a crucial role. In case $I_5$ uncertainty is considerably reduced especially when the players are enabled to use mixed strategies. In this case it is possible to create a payoff function of the zero-sum game. The unknown elements of the payoff function can be treated as some parameters which most typical values yield the lowest price for the game.

In case $I_6$ all unknown elements of payoff matrix are fuzzy variables with known membership functions. According to the definition of fuzzy set each element of the payoff function which value is unknown takes values from a definite set of numbers. These values are the elements of the corresponding fuzzy set of known reliability. The reliability function is defined on all elements of the set of numbers and maps it on numbers within the interval $[0, 1]$. In some cases, the values bringing the maximum of the reliability function can be uniquely detected. The unknown elements of the payoff matrix are to be substituted with these values. This replacement turns the partially defined antagonistic game into the classical zero-sum game with all known elements in natural way.

In case $I_7$ the solution of partially defined antagonistic game assumes an approach based on combination of above-mentioned methods of reducing partially defined antagonistic games to classical completely defined games. This combination severely depends on the unknown entries of the payoff matrix. For this case there are more then two unknown elements of the payoff matrix and these elements can be divided into several groups so that each group is represented by its own IS $I_l$, where $l = \overline{1, 6}$.

Review of possible informational situations allows to conclude the following:

1. A partially defined antagonistic game is a zero-sum matrix game with a payoff matrix containing a number of entries with unknown values.
2. One way of solving partially defined antagonistic game is based on its reduction to one or more completely defined zero-sum games. To evaluate the unknown values of the payoff matrix elements one can be use the algorithms of interpolation, extrapolation, as well as methods of pattern recognition.
3. The approach to solving partially defined antagonistic game depends on the informational situation in hand that characterizes the type and the level of uncertainty of the values of the the payoff matrix elements.
4. There are seven basic informational situations that characterize the level of uncertainty of the partially defined payoff matrix of the game.
5. The optimal solution search for the partially defined zero-sum game can contain the solutions of several completely defined zero-sum games.

## References

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