OPTIMIZATION PROBLEMS WITH PARTIAL DERIVATIVES AND ALGORITHMS FOR CONSTRUCTING GENERALIZED SOLUTIONS

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Abstract. *In the paper we define generalized solutions of the optimization problems for control systems with partial derivatives and develop two types of numerical algorithms for calculating the generalized solutions.*

We consider optimization problems of control systems that are described by the partial differential equations

$$\sum_{(k,i,j)\in K_n^1} a_{kij}^n(t,s)D^{ij}x_k(t,s) + \sum_{(k,i,j)\in K_n^2} b_{kij}^n(t,s)D^{ij}u_k(t,s) = f_n(t,s), n = \overline{1,m},$$

$$D^{ij}x_k(t,s) \triangleq \frac{\partial^{i+j_1+\ldots+j_m}x_k(t,s)}{\partial t^i \partial s^{j_1}\partial s^{j_2}\ldots\partial s^{j_m}}, D^{ij}u_k(t,s) \triangleq \frac{\partial^{i+j_1+\ldots+j_m}u_k(t,s)}{\partial t^i \partial s^{j_1}\partial s^{j_2}\ldots\partial s^{j_m}},$$

$$U \triangleq \{u(t,s) \in R^r | D^{ij}u_k(t,s) \in [u_{kij}^{\min}(t,s); u_{kij}^{\max}(t,s)], \ \ (t,s) \in D_k \subset D, \ \ (k,i,j) \in K_u\}.$$

The optimal control $u \in U$ is defined as the minimizer of the criteria functional

$$J(x,u) \triangleq \max_{l=\overline{1,L_1}} F(x,u,c^l,l)$$

under inequality constraints

$$F(x,u,d^l,l) \le g_l, \ l = \overline{L_1, L_2},$$

where

$$F(x,u,c^l,l) \triangleq \iint\limits_{D_l} \Big(\sum_{(k,i,j)\in K_l^{c1}} c_{1kij}^l(t,s)D^{ij}x_k(t,s) + \sum_{(k,i,j)\in K_l^{c2}} c_{2kij}^l(t,s)D^{ij}u_k(t,s)\Big) dtds +$$

$$+ \int\limits_{T_l} \Big(\sum_{(k,i,j)\in K_l^{c3}} c_{3kij}^l(\phi_l(\tau),\psi_l(\tau))D^{ij}x_k(\phi_l(\tau),\psi_l(\tau)) +$$

$$+ \sum_{(k,i,j)\in K_l^{c4}} c_{4kij}^l(\phi_l(\tau),\psi_l(\tau))D^{ij}u_k(\phi_l(\tau),\psi_l(\tau))\Big) d\tau +$$

$$+ \sum_{q\in Q_l} \left(\sum_{(k,i,j)\in K_l^{c5}} c_{5kij}^l(t_q^l,s_q^l)D^{ij}x_k(t_q^l,s_q^l) + \sum_{(k,i,j)\in K_l^{c6}} c_{6kij}^l(t_q^l,s_q^l)D^{ij}u_k(t_q^l,s_q^l) \right).$$

The continue differential functions c^l_{pkij}, d^l_{pkij}, $\phi_l(\tau)$, $\psi_l(\tau)$, a^n_{kij}, b^n_{kij}, f_n, and $u^{\min}_{kij}$, $u^{\max}_{kij}$ are defined on the given subsets $D \subset R^2$, $D_l \subset D$, $\{(\phi_l(\tau), \psi_l(\tau))|\tau \in T_l\} \subset D$, $\{(t^l_q, s^l_q)|q \in Q_l\} \subset D$, $K^{c_p}_l, K^{d_p}_l$, $g_l \in R$.

Numerical algorithms for optimal control approximations are based on the reductions of the primary optimal control problem to linear programming. The adequate reduction may be performed, in particular, by replacing partial derivatives $D^{ij}x_k(t,s)$ and $D^{ij}u_k(t,s)$ by correspondent difference approximations of adequate accuracy, and by implementation of appropriate numerical procedures for computing of integrals. The obtained linear programming problem is to be solved by interior point algorithms [1].

In general case of the nonlinear control systems

$$f_0(t,s,x(t,s),u(t,s),\frac{\partial x(t,s)}{\partial t},\frac{\partial u(t,s)}{\partial t},...,\frac{\partial^{\alpha^x}x(t,s)}{\partial t^{\alpha^x_t}\partial s^{\alpha^x_s}},\frac{\partial^{\alpha^u}u(t,s)}{\partial t^{\alpha^u_t}\partial s^{\alpha^u_s}})=0$$

and the nonlinear constraints

$$\iint\limits_{D_l} f_l(t,s,x(t,s),u(t,s),\frac{\partial x(t,s)}{\partial t},\frac{\partial u(t,s)}{\partial t},...,\frac{\partial^{\alpha^x}x(t,s)}{\partial t^{\alpha^x_t}\partial s^{\alpha^x_s}},\frac{\partial^{\alpha^u}u(t,s)}{\partial t^{\alpha^u_t}\partial s^{\alpha^u_s}})dsdt \le 0, l=\overline{1,n_1},$$

$$h_i(t,s,x(t,s),u(t,s),\frac{\partial x(t,s)}{\partial t},\frac{\partial u(t,s)}{\partial t},...,\frac{\partial^{\alpha^x}x(t,s)}{\partial t^{\alpha^x_t}\partial s^{\alpha^x_s}},\frac{\partial^{\alpha^u}u(t,s)}{\partial t^{\alpha^u_t}\partial s^{\alpha^u_s}}) \le 0, i=\overline{1,n_2}$$

iterative gradient methods of linearization and the modified interior point algorithms are used to built extreme controls [1, 2].

The practical example of such multidimensional optimization problem is the following inverse river pollution problem. In mathematical model of the river pollution transfer they denote by $x(t,z)$ the concentration of river water pollution at the distance coordinate z (along the river) at the time moment t. The value of the concentration $x(t,z)$ depends on concentrations$x(t,0) = u_1(t,p)$ at the initial point $z = 0$, on concentrations $x(0,z) = u_2(z,p)$ at the initial time $t = 0$, on the pollution sources intensities $u_3(t,z,p)$ at points z (industrial and agricultural production, sewage settlements, etc.), on the rate of flow $v(t,z,p)$ and on the coefficient of turbulent diffusion $a(t,z,p)$ at different points $z \in [0,b]$. These dependences are approximately described by differential equations with partial derivatives

$$\frac{\partial x(t,z)}{\partial t} = a(t,z,p)\frac{\partial^2 x(t,z)}{\partial z^2} + v(t,z,p)\frac{\partial x(t,z)}{\partial z} + u_3(t,z,p).$$

The solution of the inverse problem in search for pollution sources $u_3(t,z,p)$ is based on data measurements of concentrations $X(t_i,z_j)$ of river water contaminants at the observation points z_j, $j = 1, 2, \ldots, m$, in the time moments t_i and may be calculated as minimizer of the maximum deviation $J(u)$,

$$J(u) = \max_i \max_j |x(t_i, z_j) - X(t_i, z_j)|,$$

on the given set P of admissible parameters $p \in P$, that satisfy constraints

$$|u_1(t,p) - U_1(t)| \le C_1(t), |u_2(z,p) - U_2(z)| \le C_2(z),$$

$$|a(t,z,p) - A(t,z)| \le C_3(t,z), |v(t,z,p) - V(t,z)| \le C_4(t,z),$$

$$|f(t,z,p) - F(t,z)| \le C_5(t,z), \left|\frac{du_1(t,p)}{dt}\right| \le D_1(t), \left|\frac{du_2(z,p)}{dt}\right| \le D_2(t),$$

$$\left|\frac{\partial a(t,z,p)}{\partial t}\right| \le D_3(t,z), \left|\frac{\partial a(t,z,p)}{\partial z}\right| \le D_4(t,z), \left|\frac{\partial v(t,z,p)}{\partial t}\right| \le D_5(t,z),$$

$$\left|\frac{\partial v(t,z,p)}{\partial z}\right| \le D_6(t,z), \left|\frac{\partial f(t,z,p)}{\partial t}\right| \le D_7(t,z), \left|\frac{\partial f(t,z,p)}{\partial z}\right| \le D_8(t,z)$$

for the observed averaged values $U_1(t)$, $U_2(z)$, $A(t,z)$, $V(t,z)$, $F(t,z)$ of unknown $u_1(t,p)$, $u_2(z,p)$, $a(t,z,p)$, $v(t,z,p)$ and $f(t,z,p)$.

This inverse problem is a particular case of the general optimization problem in search for unknown functions (controls) $u : D \to R^r$ and $x : D \to R^n$, $(t,s) \in D \subset R \times R^{n_s}$, that satisfy integro-differential equations and inequalities

$$\bar{f}^k_{ij}(t,s,x,u) \triangleq f^k_{ij}(t,s,u(t,s),F^{f^k_{ij}}(x,t,s)) = 0, (t,s) \in D^i_j(x,u), k = \overline{1,k_{ij}},$$

$$\bar{g}^l_{ij}(t,s,x,u) \triangleq g^l_{ij}(t,s,u(t,s),F^{g^l_{ij}}(x,t,s)) \le 0, (t,s) \in D^i_j(x,u), l = \overline{1,l_{ij}},$$

where f^k_{ij} and g^k_{ij} are given functions on given subsets $D^i_j(x,u) \subset D$, $j = \overline{1,m+1}$, $D^i_0(x,u) \triangleq \{t^i_q(x,u), s^i_q(x,u)\}^{q_i}_{q=1} \subset D$, $i = \overline{1,i_j}$; $F^{f^k_{ij}}$ and $F^{g^k_{ij}}$ are given compositions of operators F_1, F_2 and F_3:

$$F_1(x,t,s,\alpha,\beta) \triangleq (x(t,s), \frac{\partial}{\partial t}x(t,s), \frac{\partial}{\partial s}x(t,s), ..., \frac{\partial^{\alpha+\beta}}{\partial t^\alpha \partial s^\beta}x(t,s)),$$

F_2 is defined by the set $\Omega(t,s) \triangleq \{t^i(t,s), s^i(t,s), \alpha^i, \beta^i\}^{n_\Omega}_{i=1}$,

$$F_2(F_1,x,t,s,\Omega) \triangleq (F_1(x,t+t^1(x,t),s+s^1(x,t),\alpha^1,\beta^1),$$

$$F_1(x, t+t^2(x,t), s+s^2(x,t), \alpha^2, \beta^2), \ldots, F_1(x, t+t^{n\Omega}(x,t), s+s^{n\Omega}(x,t), \alpha^{n\Omega}, \beta^{n\Omega})) =$$

$$= (x(t+t^1(t,s), s+s^1(t,s)), ..., \frac{\partial^{\alpha^1+\beta^1}}{\partial t^{\alpha^1} \partial s^{\beta^1}} x(t+t^1(t,s), s+s^1(t,s)),$$

$$x(t+t^2(t,s), s+s^2(t,s)), ..., \frac{\partial^{\alpha^2+\beta^2}}{\partial t^{\alpha^2} \partial s^{\beta^2}} x(t+t^2(t,s), s+s^2(t,s)), ...,$$

$$x(t+t^{n\Omega}, s+s^{n\Omega}), \frac{\partial}{\partial t} x(t+t^{n\Omega}, s+s^{n\Omega}), ..., \frac{\partial^{\alpha^{n\Omega}+\beta^{n\Omega}}}{\partial t^{\alpha^{n\Omega}} \partial s^{\beta^{n\Omega}}} x(t+t^{n\Omega}, s+s^{n\Omega})))$$

and F_3 is defined by the given operator ϕ on the given set $\tilde{\Omega}(t,s,x,u) \subset R \times R^{n_s}$,

$$F_3(x,u,t,s,\phi,\tilde{\Omega}) \triangleq \iint\limits_{\tilde{\Omega}(t,s,x,u)} \phi(t,s,u(t,s), F_1(x, t+\tau, s+\sigma, \alpha, \beta)) d\tau d\sigma.$$

In search for extremal solution of such generalized optimization problem we may implement subgradient methods. In case of convex functions one use generalized gradient algorithms to calculate approximated global optimal solutions. In this way the parameter set $\Omega(\alpha_r)$ of all the functions (x,u), that satisfy the inequalities

$$\bar{f}^k_{ij}(t,s,x,u) \leq \alpha_r, \ \ \bar{h}^k_{ij}(t,s,x,u) \leq \alpha_r, (t,s) \in D^i_j(x,u), k = \overline{1,k_{ij}},$$
$$\bar{g}^l_{ij}(t,s,x,u) \leq 0, (t,s) \in D^i_j(x,u), l = \overline{1,l_{ij}},$$
$$\bar{h}^k_{ij} \triangleq -\bar{f}^k_{ij}, j = \overline{0,m+1}, i = \overline{1,i_j}$$

is defined and the generalized solution is defined as a subsequence of the sequence $\{(x_r,u_r)\}_{r=1}^{\infty} \in \Omega(\alpha_r)$, that satisfy the inequalities $B(x_k,u_k) \leq \inf\limits_{(x,u)\in\Omega(\alpha_r)} B(x,u) + \alpha_r$ at $\alpha_r \to 0$. The generalized solution is to be calculated by numerical methods [1,2] as a sequence of functions $(x_r(t,s), u_r(t,s))$, belonging to nested sets $X^{n_x(r)} \subset X^{n_x(r)+1}$, $U^{n_u(r)} \subset U^{n_u(r)+1}$ of parametric functions

$$(x_r(t,s), u_r(t,s)) \triangleq (x_{n_x(r)}(p_r,t,s), u_{n_u(r)}(q_r,t,s)) \in X^{n_x(r)} \times U^{n_u(r)}$$

that are defined by the parameters $p_r \in R^{n_x(r)}$, $q_r \in R^{n_u(r)}$, where for any value $\alpha > 0$ there exists a number r for which the parameters p_r, q_r satisfy the inequalities

$$\max_{(t,s)\in D^i_j(x,u)} \bar{f}^k_{ij}(t,s,x_{n_x(r)}(p_r,\cdot,\cdot), u_{n_u(r)}(q_r,\cdot,\cdot)) \leq \alpha, k = \overline{1,k_{ij}}$$

$$\max_{(t,s)\in D^i_j(x,u)} \bar{h}^k_{ij}(t,s,x_{n_x(r)}(p_k,\cdot,\cdot), u_{n_u(r)}(q_k,\cdot,\cdot)) \leq \alpha, \ k = \overline{1,k_{ij}}, \tag{1}$$

$$\max_{(t,s)\in D_j^i(x,u)} \bar{g}_{ij}^l(t,s,x_{n_x(r)}(p_k,\cdot,\cdot),u_{n_u(r)}(q_k,\cdot,\cdot)) \le 0, l=\overline{1,l_{ij}},$$

$$B(x_{n_x(r)}(p_k,\cdot,\cdot),u_{n_u(r)}(q_k,\cdot,\cdot)) \le \inf_{(x,u)\in\Omega(\alpha)} B(x,u)+\alpha.$$

Numerical algorithms for calculating generalized solutions are given by the following theorem.

Theorem 1. *If for each $\alpha > 0$ and for selected sequence of nested setsX^r, U^r, $r=\overline{1,\infty}$, convex on (p,q) functionals*

$$B(x_r(p,\cdot,\cdot),u_r(q,\cdot,\cdot)), \bar{g}_{ij}^k(t,s,x_r(p,\cdot,\cdot),u_r(q,\cdot,\cdot))$$

and for the linear functionals

$$\bar{f}_{ij}^k(t,s,x_r(p,\cdot,\cdot),u_r(q,\cdot,\cdot)), k=\overline{1,k_{ij}}$$

there exists a number r, for which the set of parameters $p\in R^r$ and $q\in R^r$, which satisfy the inequalities (1), has an open subset, then the generalized solution is contained in the sequence $\{x_{n_x(r)}(p_r,\cdot,\cdot),u_{n_u(r)}(q_r,\cdot,\cdot)\}_{r=2}^{\infty}$ and is calculated by the iterative algorithm:

$$p_{r+1}=p_r-h_r v_r/||v_r||, q_{r+1}=q_r-h_r w_r/||w_r||,$$

$$(v_r,w_r)=\begin{array}{r} \bar{\nabla}_{(p,q)}\bar{f}_{ij}^k(t,s,x_{n_x(r)}(p_r,\cdot,\cdot),u_{n_u(r)}(q_r,\cdot,\cdot)),\ if \\ \bar{f}_{ij}^k(t,s,x_{n_x(r)}(p_r,\cdot,\cdot),u_{n_u(r)}(q_r,\cdot,\cdot))=z, \\ \bar{\nabla}_{(p,q)}\bar{h}_{ij}^k(t,s,x_{n_x(r)}(p_r,\cdot,\cdot),u_{n_u(r)}(q_r,\cdot,\cdot)),\ if \\ \bar{h}_{ij}^k(t,s,x_{n_x(r)}(p_r,\cdot,\cdot),u_{n_u(r)}(q_r,\cdot,\cdot))=z, \\ \bar{\nabla}_{(p,q)}\bar{g}_{ij}^k(t,s,x_{n_x(r)}(p_r,\cdot,\cdot),u_{n_u(r)}(q_r,\cdot,\cdot)),\ if \\ \bar{g}_{ij}^k(t,s,x_{n_x(r)}(p_r,\cdot,\cdot),u_{n_u(r)}(q_r,\cdot,\cdot))=z, \\ \bar{\nabla}_{(p,q)}f_0(x_{n_x(r)}(p_r,\cdot,\cdot),u_{n_u(r)}(q_r,\cdot,\cdot)),\ if\ z\le 0, \end{array}$$

$$z=\max\{\max_{j=\overline{0,m+1}}\max_{i=\overline{1,i_j}}\max_{k=\overline{1,k_{ij}}}\max_{(t,s)\in D_j^i(x,u)}\bar{f}_{ij}^k(t,s,x_{n_x(r)}(p_r,\cdot,\cdot),u_{n_u(r)}(q_r,\cdot,\cdot)),$$
$$\max_{j=\overline{0,m+1}}\max_{i=\overline{1,i_j}}\max_{k=\overline{1,k_{ij}}}\max_{(t,s)\in D_j^i(x,u)}\bar{h}_{ij}^k(t,s,x_{n_x(r)}(p_r,\cdot,\cdot),u_{n_u(r)}(q_r,\cdot,\cdot)),$$
$$\max_{j=\overline{0,m+1}}\max_{i=\overline{1,i_j}}\max_{k=\overline{1,k_{ij}}}\max_{(t,s)\in D_j^i(x,u)}\bar{g}_{ij}^k(t,s,x_{n_x(r)}(p_r,\cdot,\cdot),u_{n_u(r)}(q_r,\cdot,\cdot))\},$$

$$\lim_{r\to\infty}h_r=0, \lim_{r\to\infty}n_x(r)=\infty, \lim_{r\to\infty}n_u(r)=\infty, \sum_{r=1}^{\infty}h_r=\infty, h_r>0.$$

In case of linear optimization problems the optimal solution may be calculated by accelerated algorithms using interior point methods. In this way the original generalized optimization problem is approximated by the LP problem

$$\min c^T x | Ax = b, x \geq 0$$

that is solved simultaneously with the dual problem

$$\max b^T y \mid A^T y + z = c, \ z \geq 0.$$

By the Karush-Kuhn-Tucker theorem the solution of these LP is the solutions of the nonlinear system (and backwards)

$$\begin{gathered} Ax - b = 0, \ A^T y + z - c = 0, \ ZXe = 0, x \geq 0, \ z \geq 0, \\ e = (1, 1, ..., 1), \ X = diag(x), Z = diag(z). \end{gathered}$$

To calculate the solution (x, y, z) of the last nonlinear system the Newton's iterative methods may be effectively implemented starting from any interior admissible point $(x^0, y^0, z^0), x^0 > 0, \ z^0 > 0$. At the k-th iteration the solution $(\delta x, \delta y, \delta z)$ of the linearized at the point $\left(x^k, y^k, z^k\right)$ Newton system

$$\begin{gathered} A\delta x = r_p, A^T \delta y + \delta z = r_d, Z^k \delta x + X^k \delta z = r_a, \\ r_p = b - Ax^k, r_d = c - z^k - A^T y^k, r_a = -X^k Z^k e \end{gathered}$$

is calculated

$$! = \left(X^{-1} Z\right)^{-1}, \ ACA^T \delta y = r_p + C(r_d - X^{-1} r_a), \delta x = CA^T \delta y - C(r_d - X^{-1} r_a), \ \delta z = r_d - A^T \delta y$$

To ensure the inequalities $x^{k+1} > 0, \ z^{k+1} > 0$ we calculate

$$\alpha_1 = \min_i \left(\frac{-x_i}{\delta x_i}\right) | \delta x_i < 0 \ , \alpha_2 = \min_i \left(\frac{-z_i}{\delta z_i}\right) | \delta z_i < 0 \ ,$$

$$\tilde{\alpha} = \min \{\alpha_1, \alpha_2\}, \gamma^k = \left(x^k\right)^T z^k, \tilde{\gamma}^k = \left(x^k + \tilde{\alpha}^k \delta x^k\right)^T \left(z^k + \tilde{\alpha}^k \delta z^k\right),$$

$$\sigma^k = \left(\frac{\tilde{\gamma}^k}{\gamma^k}\right)^2, \mu^k = \sigma^k \left(\frac{\gamma^k}{n}\right),$$

$$r_a = \mu^k e - \Delta_a X^k \Delta_a Z^k e - X^k Z^k e, \ \Delta_a X^k = diag(\delta x), \ \Delta_a Z^k = diag(\delta z),$$

$$\begin{gathered} ACA^T \Delta y = r_p + C(r_d - X^{-1} r_a), \\ \Delta x = CA^T \Delta y - C(r_d - X^{-1} r_a), \Delta z = r_d - A^T \Delta y, \\ \left(x^{k+1}, y^{k+1}, z^{k+1}\right) = \left(x^k, y^k, z^k\right) + \alpha \left(\Delta x, \Delta y, \Delta z\right). \end{gathered}$$

The approximate solution is obtained at the iteration satisfying the three inequalities $||\Delta x|| < e, ||\Delta y|| < e, ||\Delta z|| < e$. In general case of regular convex optimization problem the polynomial convergence of this algorithm was proved.

Conclusions

Two types of numerical algorithms for calculating the generalized solutions of the generalized optimization control systems with partial derivatives is proposed: the gradient algorithm for calculating extremal solutions and the Newton type interior point algorithm for calculating the global optimal generalized solutions of linear control systems.

References

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